تمرین سری 11

1)

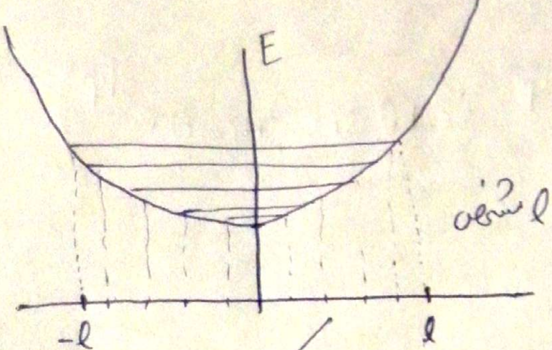
الف)

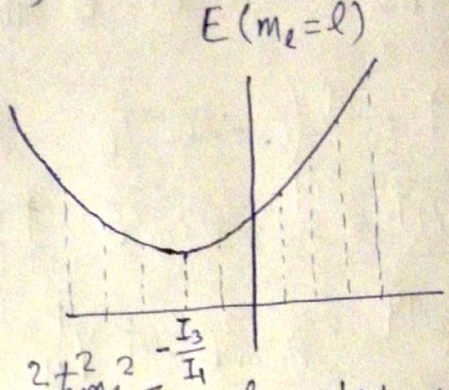
$$H=\frac{1}{2I_1}(L^2-L_z^2)+\frac{1}{2I_3}L_z^2=\frac{1}{2I_1}\left(L^2+\left(\frac{I_3}{I_1}-1\right)L_z^2\right)$$

با اثر دادن آن روی حالت $|l,m\rangle$ داریم:

$$H|l,m\rangle=\frac{\hbar^2}{2I_1}\left(l(l+1)+\left(\frac{I_1}{I_3}-1\right)m_l^2\right)$$

ب)





پ) در این حالت انرژی ها برای هر l به صورت $E_l=\frac{\hbar^2}{2I_1}\frac{I_1}{I_3}m_l^2=\frac{\hbar^2 m_l^2}{2I_3}$ هستند $-l<m_l<l$ است.

ت)

$$H=\frac{L^2}{2I}+\alpha L_z \Rightarrow E_{l,m}=\hbar^2\left(\frac{l(l+1)}{2I}+\alpha m\right)$$

2)

$$xy=r^2\sin^2\theta\sin\varphi\cos\varphi=\frac{r^2}{2}\sin^2\theta\sin(2\varphi)$$

$$yz=r^2\sin\theta\cos\theta\sin\varphi \qquad \sin\theta\cos\theta\times\sin\varphi=\sqrt{\frac{15}{2\pi}}^{-1}\left(-Y_2^1-Y_2^{-1}\right)$$

$$xz=r^2\sin\theta\cos\theta\cos\varphi \qquad \sin\theta\cos\theta\cos\varphi=\sqrt{\frac{15}{2\pi}}^{-1}\left(-Y_2^1+Y_2^{-1}\right)$$

$$\sin^2\theta\sin\varphi\cos\varphi=\left(\frac{1}{2}\sqrt{\frac{15}{2\pi}}\right)^{-1}\left(Y_2^2-Y_2^{-2}\right)\times\frac{1}{2}$$

$$\Rightarrow (xy+yz+xz)=\sqrt{\frac{2\pi}{15}}\left(-2Y_2^1-Y_2^{-2}+Y_2^2\right)$$

$$\Rightarrow \psi=C\sqrt{\frac{2\pi}{15}}\left(Y_2^2-Y_2^{-2}-2Y_2^1\right)r^2e^{-\alpha r^2}$$

بدلیل اینکه ψ هیچ مولفه ای در راستای Y_0^0 ندارد پس احتمال اندازه گیری $l=0$ برابر صفر است. اما احتمال اندازه گیری $l=2$ $(L^2=6\hbar^2)$ برابر:

$$\sum_m|\langle 2,m|\psi\rangle|^2=\frac{2\pi}{15}C^2(1+0+0+4+1)=\frac{2\pi}{15}C^2\times 6$$

اما از بهنجارش ψ داریم:

$$\langle Y_l^m,Y_{l'}^{m'}\rangle=\frac{4\pi}{2l+1}\delta_{ll'}\delta_{mm'}$$

$$1=\frac{4\pi}{5}\times\frac{2\pi}{15}C^2\times 6$$

$$\Rightarrow \frac{2\pi}{15}C^2\times 6=\frac{5}{4\pi}$$

$$\Rightarrow P(L^2=6\hbar^2)=\frac{5}{4\pi}$$

احتمال نسبی مقادیر $m_\ell=-2:2$ برابر ۱:۰:۰:۴:۱ است.

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$$Y_{44} = Ae^{4i\varphi}\sin^4\theta \qquad L_- = -\hbar e^{-i\varphi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\varphi}\right)$$

$$L_-Y_4^4 = -A\hbar e^{-i\varphi}\left(4\sin^3\theta\cos\theta\, e^{4i\varphi} - i\cot\theta\, 4i e^{4i\varphi}\sin^4\theta\right)$$

$$= -8A\hbar e^{3i\varphi}\sin^3\theta\cos\theta = \hbar Y_4^3$$

$$L_-Y_4^3 = 8A\hbar e^{-i\varphi}\left((3\sin^2\theta\cos^2\theta - \sin^4\theta)e^{3i\varphi} - i\cot\theta\, 3i\sin^3\theta\cos\theta\, e^{3i\varphi}\right)$$

$$= 24A\hbar e^{2i\varphi}\sin^2\theta\cos^2\theta = \hbar Y_4^2$$

$$L_-Y_4^2 \sim e^{i\varphi}\sin\theta\cos^3\theta = \hbar Y_4^1$$

$$L_-Y_4^1 \sim \cos^4\theta = \hbar Y_4^0$$

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$$\begin{cases} x = r\sin\theta\cos\varphi = r\sqrt{\frac{2\pi}{3}}\left(-Y_1^1 - Y_1^{-1}\right) \\ z = r\cos\theta = r\,2\sqrt{\frac{\pi}{3}}\,Y_1^0 \end{cases}$$

$$\psi = \frac{1}{r}\left\{\frac{1}{2\sqrt3}Y_1^0 + \frac{\sqrt2}{3}\left(-Y_1^1 - Y_1^{-1}\right)\right\}$$

$$\Rightarrow L^2\psi \sim \frac{\hbar^2}{2\sqrt3}1\times2Y_1^0 + \frac{\sqrt2}{3}\left(-1\times2Y_1^1 - 1\times2Y_1^{-1}\right) = \left(-\frac{4\sqrt2}{3} + \frac{1}{\sqrt3}\right)\hbar^2 \dots = 2\hbar^2\psi$$

$$L_z\psi \sim \frac{1}{2\sqrt3}(0)Y_1^0 + \frac{\sqrt2}{3}\left(-1\,Y_1^1 + 1\,Y_1^{-1}\right)$$

$+:$

$-:$ $$L_\pm\psi = \frac{1}{r}\left\{\frac{1}{2\sqrt3}\begin{pmatrix}Y_1^1\\ Y_1^{-1}\end{pmatrix} \mp \frac{\sqrt2}{3}\begin{pmatrix}0 & -Y_1^0\\ +Y_1^0 & -0\end{pmatrix}\right\}\hbar = \frac{1}{r}\left\{\frac{1}{2\sqrt3}Y_1^{\pm1} \pm \frac{\sqrt2}{3}Y_1^0\right\}\hbar$$

و ویژه مقدار L^2 است.

$$\langle L_\pm\rangle = \hbar\left\{\frac{1}{2\sqrt3}\cancel{\langle Y_1^0, Y_1^{\pm1}\rangle} \pm \frac{\sqrt2}{2\sqrt3}\langle Y_1^0, Y_1^0\rangle + \frac{\sqrt2}{2\times3\sqrt3}\left(-\langle Y_1^1, Y_1^{\pm1}\rangle + 0\right)\right.$$

$$\left. - \frac{\sqrt2}{3}\langle Y_1^{-1}, Y_1^{\pm1}\rangle\frac{1}{2\sqrt3} \mp \frac{2}{9}\cancel{\langle Y_1^{-1}, Y_1^0\rangle}^{0}\right\}$$

$$= \hbar\left\{\pm\frac{\sqrt2}{2\sqrt3}\frac{4\pi}{3} - \frac{\sqrt2}{6\sqrt3}\right\} = \hbar\left(\pm\left(\frac{2}{3}\right)^{3/2}\pi - \frac{\sqrt2}{6\sqrt3}\right)$$