تمرین سری دهم

۱. نمایش ماتریسی

الف)

$a|n\rangle = \sqrt{n}\,|n-1\rangle \qquad a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$

$\langle n|m\rangle = \delta_{nm}$

$$a = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{4} \\ & & & & & \ddots \end{pmatrix}$$

$$a^+ = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \sqrt{1} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{4} & 0 \\ & & & & & \ddots \end{pmatrix}$$

ب)

$$X = \sqrt{\frac{\hbar}{2m\omega}}\,(a+a^+) = \sqrt{\frac{\hbar}{2m\omega}} \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 \\ \sqrt{1} & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} \\ & & & & \ddots \end{pmatrix}$$

$$X^2 = \frac{\hbar}{2m\omega} \begin{pmatrix} 1 & 0 & \sqrt{1\times 2} & 0 & 0 \\ 0 & 1+2 & 0 & \sqrt{3\times 2} & 0 \\ \sqrt{1\times 2} & 0 & 2+3 & 0 & \sqrt{3\times 4} \\ 0 & \sqrt{2\times 3} & 0 & 3+4 & 0 \\ 0 & 0 & \sqrt{3\times 4} & 0 & 4+5 \\ & & & & & \ddots \end{pmatrix}$$

پ)

$$P = i\sqrt{\frac{m\hbar\omega}{2}}\,(a^{\dagger} - a) = i\sqrt{\frac{m\hbar\omega}{2}} \begin{pmatrix} 0 & -\sqrt{1} & 0 & 0 & 0 \\ \sqrt{1} & 0 & -\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 & -\sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} & 0 & -\sqrt{4} \\ 0 & 0 & 0 & \sqrt{4} & 0 \\ & & & & & \ddots \end{pmatrix}$$

$$P^2 = \frac{m\hbar\omega}{2} \begin{pmatrix} 1 & 0 & -\sqrt{1\times 2} & 0 & 0 \\ 0 & 1+2 & 0 & -\sqrt{2\times 3} & 0 \\ -\sqrt{1\times 2} & 0 & 2+3 & 0 & -\sqrt{3\times 4} \\ 0 & -\sqrt{2\times 3} & 0 & 3+4 & 0 \\ 0 & 0 & -\sqrt{3\times 4} & 0 & 4+5 \\ & & & & & \ddots \end{pmatrix}$$

ت) با توجه به قسمت های قبل و این که $H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$ می‌توان نمایش H را یافت

ولی راه ساده تری هم وجود دارد.

$$H|n\rangle = \hbar\omega\left(n + \tfrac{1}{2}\right)|n\rangle$$

$$H = \hbar\omega \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 3/2 & 0 & 0 \\ 0 & 0 & 5/2 & 0 \\ 0 & 0 & 0 & 7/2 \\ & & & & \ddots \end{pmatrix}$$

2. الف)

$$|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0\rangle$$

$$\hat{a}|n\rangle = \frac{1}{\sqrt{n!}}\hat{a}\left[(\hat{a}^\dagger)^n|0\rangle\right] = \frac{1}{\sqrt{n!}} n(\hat{a}^\dagger)^{n-1}|0\rangle = \sqrt{n}\,\frac{(\hat{a}^\dagger)^{n-1}}{\sqrt{(n-1)!}}|0\rangle = \sqrt{n}\,|n-1\rangle$$

use eq. 6-47
from the book

ب)

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^\dagger) \qquad \hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}^\dagger-\hat{a})$$

$$\langle m|\hat{x}|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle m|a+a^\dagger|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\langle m|n-1\rangle + \sqrt{m}\langle m-1|n\rangle\right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n}\,\delta_{m,n-1} + \sqrt{n+1}\,\delta_{m,n+1}\right]$$

پ)

$$\langle m|\hat{p}|n\rangle = i\sqrt{\frac{m\omega\hbar}{2}}\langle m|a^\dagger - a|n\rangle = i\sqrt{\frac{m\omega\hbar}{2}}\left[\sqrt{n+1}\,\delta_{m,n+1} - \sqrt{n}\,\delta_{m,n-1}\right]$$

ت)

$$\langle m|\hat{p}\hat{x}|n\rangle = \langle m|\hat{p}\mathbb{1}\hat{x}|n\rangle = \sum_k \langle m|\hat{p}|k\rangle\langle k|\hat{x}|n\rangle \qquad \hat{\mathbb{1}} = \sum_k |k\rangle\langle k|$$

$$= \frac{i\hbar}{2}\sum_k\left(\sqrt{m}\,\delta_{m-1,k} - \sqrt{m+1}\,\delta_{m+1,k}\right)\left(\sqrt{n}\,\delta_{k,n-1} + \sqrt{n+1}\,\delta_{k,n+1}\right)$$

$$= \frac{i\hbar}{2}\left[\sqrt{mn}\,\delta_{m,n} - \sqrt{(m+1)n}\,\delta_{m+1,n-1} + \sqrt{m(n+1)}\,\delta_{m-1,n+1} - \sqrt{(m+1)(n+1)}\,\delta_{m+1,n+1}\right]$$

$$= \frac{i\hbar}{2}\left[-\delta_{m,n} - \sqrt{(m+1)(m+2)}\,\delta_{m+2,n} + \sqrt{(n+1)(n+2)}\,\delta_{m,n+2}\right]$$

since $\hat{x}^\dagger = x$ and $\hat{p}^\dagger = \hat{p}$:

$$\langle m|\hat{x}\hat{p}|n\rangle = \left[(\langle m|\hat{x}\hat{p}|n\rangle)^*\right]^* = \left[\langle n|(\hat{x}\hat{p})^\dagger|m\rangle\right]^* = \left[\langle n|\hat{p}\hat{x}|m\rangle\right]^*$$

$$= -\frac{i\hbar}{2}\left[-\delta_{m,n} - \sqrt{(n+1)(n+2)}\,\delta_{m,n+2} + \sqrt{(m+1)(m+2)}\,\delta_{n,m+2}\right]$$

ث)

$$\langle m|[\hat{x},\hat{p}]|n\rangle = \langle m|\hat{x}\hat{p} - \hat{p}\hat{x}|n\rangle = \langle m|\hat{x}\hat{p}|n\rangle - \langle m|\hat{p}\hat{x}|n\rangle$$

$$= i\hbar\,\delta_{mn}$$

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Eigen functions are :

$$\langle x | T_\xi^{-1} | n \rangle = \langle x - \xi | n \rangle = \phi_n(x-\xi)$$

$$\Delta x^2 = \langle n | \hat{x}^2 | n \rangle - \underbrace{\langle n | \hat{x} | n \rangle^2}_{\text{zero}} \Rightarrow \Delta x = \sqrt{\langle n | \hat{x}^2 | n \rangle}$$ (2

As the same $\Delta p = \sqrt{\langle n | \hat{p}^2 | n \rangle}$

$$\hat{x}^2 = \frac{\hbar}{2m\omega}\left(\hat{a} + \hat{a}^\dagger\right)^2 = \frac{\hbar}{2m\omega}(\hat{a}+\hat{a}^\dagger)(\hat{a}+\hat{a}^\dagger) = \frac{\hbar}{2m\omega}\left[\hat{a}^2 + \hat{a}^{\dagger 2} + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right]$$

Since $[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = \hat{\mathbb{1}}$

$$\hat{x}^2 = \frac{\hbar}{2m\omega}\left[\hat{a}^2 + \hat{a}^{\dagger 2} + 2\hat{a}^\dagger a + \hat{\mathbb{1}}\right]$$

$$\hat{p}^2 = -\frac{m\hbar\omega}{2}\left[\hat{a}^2 + \hat{a}^{\dagger 2} - 2\hat{a}^\dagger\hat{a} - \hat{\mathbb{1}}\right]$$

$$\langle n | \hat{x}^2 | n \rangle = \frac{\hbar}{2m\omega}\Big[\underbrace{\langle n | \hat{a}^2 | n \rangle}_{\text{zero}} + \underbrace{\langle n | \hat{a}^{\dagger 2} | n \rangle}_{\text{zero}} + 2\underbrace{\langle n | a^\dagger a | n \rangle}_{n} + \underbrace{\langle n | \hat{\mathbb{1}} | n \rangle}_{1}\Big]$$

$$= \frac{\hbar}{2m\omega}(2n+1)$$

$$\langle n | \hat{p}^2 | n \rangle = \frac{m\omega\hbar}{2}(2n+1)$$

$$\Delta x = \sqrt{\frac{\hbar}{2m\omega}(2n+1)} \qquad \Delta p = \sqrt{\frac{m\omega\hbar}{2}(2n+1)}$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 - eE\hat{x} = \hat{H}_0 - eE\hat{x}$$ .3

$$= \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\left[\hat{x} - \frac{eE}{m\omega^2}\right]^2 - \frac{e^2E^2}{2m\omega^2}$$

$$E_0 \equiv \frac{e^2E^2}{2m\omega^2} \qquad \xi \equiv \frac{eE}{m\omega^2}$$

$$\Rightarrow \hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\left[\hat{x} - \xi\right]^2 - E_0$$

$$\hat{T}_\xi = e^{-\frac{i}{\hbar}\xi\hat{p}} \qquad \hat{T}_\xi|x\rangle = |x-\xi\rangle \qquad \hat{T}_\xi^{-1}\hat{x}\,T_\xi^{-1} = \hat{x} - \xi$$

$$\Rightarrow \hat{H} = T_\xi^{-1} H_0 T_\xi - E_0$$

$$\hat{H}|\psi_n\rangle = T_\xi^{-1} H_0 \underbrace{T_\xi T_\xi^{-1}}_{\mathbb{1}}|n\rangle - E_0|\psi_n\rangle = T_\xi^{-1}\underbrace{H_0|n\rangle}_{E_n|n\rangle} - E_0|\psi_n\rangle = (E_n - E_0)|\psi_n\rangle$$

$$= \left[\hbar\omega\left(n+\tfrac{1}{2}\right) - E_0\right]|\psi_n\rangle$$

Eigen functions are:

$$\psi_n(x) = \langle x|\psi_n\rangle = \langle x|T_\xi^{-1}|n\rangle = \langle x-\xi|n\rangle = \phi_n(x-\xi)$$

$$E_n = \hbar\omega(n+1/2)$$

The classical turning points occur at: 4

$$E_n = \frac{1}{2}m\omega^2\tilde{x}_n^2 \qquad \tilde{x}_n = \sqrt{\frac{\hbar}{m\omega}(2n+1)}$$

$$P_n(\text{outside}) = 1 - P(\text{inside}) = 1 - \int_{-\tilde{x}_n}^{\tilde{x}_n} |\phi_n(x)|^2 dx$$

$$n=0 \Rightarrow P_0(\text{outside}) = 1 - \int_{-\tilde{x}_0}^{\tilde{x}_0} \frac{1}{\sqrt{\pi}} e^{-\frac{m\omega x^2}{\hbar}} dx \qquad t \equiv \sqrt{\frac{m\omega}{\hbar}}x$$

$$= 1 - \frac{\tilde{x}_0^2}{\sqrt{\pi}} \int_{-1}^{1} e^{-t^2} dt$$

$$= 1 - \frac{2}{\sqrt{\pi}}\left(\frac{\hbar}{m\omega}\right) \int_0^1 e^{-t^2} dt$$

$$= 1 - \frac{\sqrt{\hbar}}{m\omega} \operatorname{erf}(1)$$

where $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2} dt$.

$$n=1 \Rightarrow P_1(\text{outside}) = 1 - \int_{-\tilde{x}_1}^{\tilde{x}_1} \frac{1}{\sqrt{\pi}} \frac{m\omega}{\hbar} x^2 e^{-\frac{m\omega x^2}{\hbar}} dx \qquad \alpha \equiv \frac{m\omega}{\hbar}$$

$$= 1 - \frac{2}{\sqrt{\pi}}\alpha \int_0^{\tilde{x}_1} x^2 e^{-\alpha x^2} dx$$

$$= 1 - \frac{2\alpha}{\sqrt{\pi}} \frac{-\partial}{\partial\alpha}\left[\int_0^{\tilde{x}_1} e^{-\alpha x^2} dx\right]$$

$$P_1(\text{outside}) = 1 + \frac{2\alpha}{\sqrt{\pi}} \frac{\partial}{\partial\alpha}\left[\int_0^{\sqrt{3\hbar/m\omega}} e^{-\alpha x^2}\, dx\right] \qquad t \equiv \sqrt{\alpha}\, x$$

$$= 1 + \frac{2\alpha}{\sqrt{\pi}} \frac{\partial}{\partial\alpha}\left[\int_0^{\sqrt{3}} e^{-t^2}\, dt \, \frac{1}{\sqrt{\alpha}}\right]$$

$$= 1 + \alpha \frac{\partial}{\partial\alpha}\left(\frac{1}{\sqrt{\alpha}}\right) \times \operatorname{erf}(\sqrt{3})$$

$$= 1 + \alpha\left(-\frac{1}{2\alpha\sqrt{\alpha}}\right) \operatorname{erf}(\sqrt{3})$$

$$P_1(\text{outside}) = 1 - \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}\, \operatorname{erf}(\sqrt{3})$$

$n=2 \Rightarrow$ This is like previous cases with a little bit more calculations.

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