An Introduction to Wavelet Transform (WT)

Advanced Structural Dynamics

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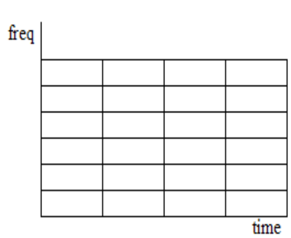
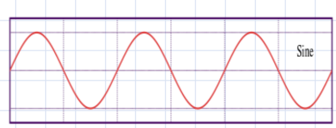
Scope and Goals

- Need a suitable transform providing suitable time and frequency resolutions.
- Provide us with the frequency of the signals and the time associated to those frequencies
- May need different time resolutions for various ranges of frequency.
  - e.g. higher frequencies require higher time resolutions and vice versa.

Introduction

- From STFT to WT
  - Short Time Fourier Transform(STFT) has noticeable defects such as:
    - Constant time-frequency resolution over all frequencies
    - Usage of sinusoid basis functions which extends from minus infinity to plus infinity





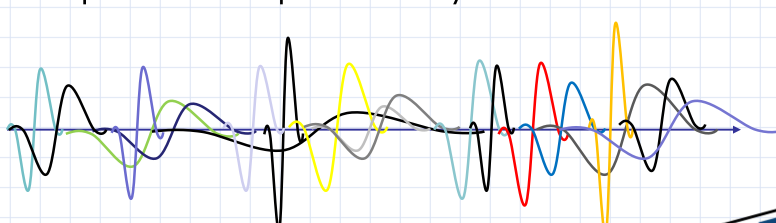
Introduction

- More robust convolution-based transforms have been developed for signal processing that do not have the deficiencies of STFT.
- **Wavelet Transform** (WT) has emerged as a powerful tool for time-scale representation of signals.
- A **Wavelet Series** is a representation of a square-integrable (real- or complex-valued) function by a certain orthonormal series generated by a wavelet.
- That is, using WT the input signal will be decomposed into a series of scaled wavelets that may occur at different times.

Introduction

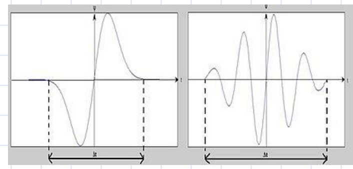
- Schematically, given a signal and a wavelet:

- The simplified decomposition may look like this:



Introduction

- A wavelet is a wave-like oscillation with an amplitude that begins at zero, increases, and then decreases back to zero.



- In this context, scale (on time axis) is the reverse of frequency, such that:

Large scale → **Low frequency**

Small scale → **High frequency**

Introduction

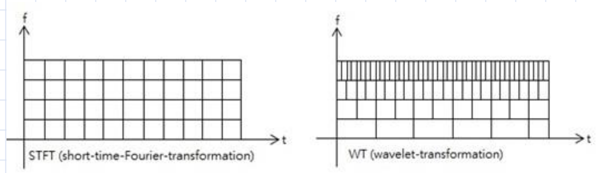
- Scale gives us some great capabilities such as better resolution, possibility of time-dependent denoising, etc.

- WT is a linear transform – so it is very well fitted for civil engineering applications like modal decomposition.

- However, compared to FT and STFT, WT needs a rather strong mathematical background.

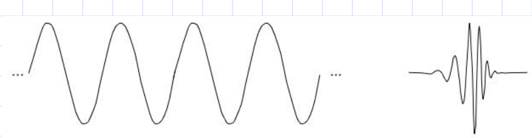
Introduction

- A perfect transform in terms of time-frequency resolution:
  - With additional special properties of the wavelets, the resolution in time at higher analysis frequencies of the basis function is improved.
  - As a result, WT has better frequency resolution in low frequencies (high scales), while at high frequencies (low scales) the time resolution is more precise, as needed in most cases.



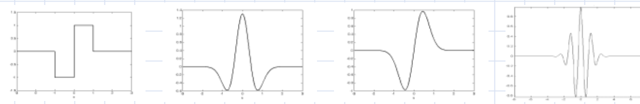
Wavelets

- WT is performed by usage of "Compactly Supported" functions known as "Mother Wavelets" as basis function rather than infinite sinusoids in FT.
- A function has compact support if it is zero outside of a subset of its domain of definition. One meaning of compact support is that by compacting the domain, we increase the possibility of victory. Compactly supported functions refer to functions that in both time and frequency domains have finite lengths.

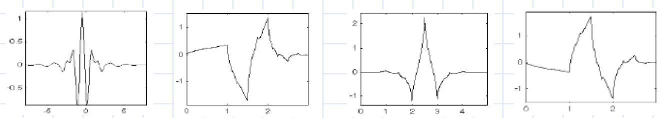


Wavelets

- Some Mother Wavelets



Haar | Mexican hat | PDF's Derivative | Morlet



Mayer | Symlet | Coiflet | Daubechies

- Selection of mother wavelet is usually based on input signal and the application of the transform.

Wavelets

- A function could be regarded as wavelet provided that:

$$\int_{-\infty}^{\infty} \psi(t)dt = 0$$

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\Psi(\omega)|^2}{|\omega|} d\omega < +\infty \quad \textit{Admissibility Condition}$$

$$|\Psi(\omega)|^2\Big|_{\omega=0} = 0$$

- Mother wavelet has zero average in time-domain and in order to satisfy admissibility condition, wavelet function at zero frequency is equal to zero. this means that any wavelet function is a band-pass filter in frequency domain.

Comparison of WT with STFT

- STFT has constant time-frequency resolution, while WT has variable-window resolution which is best fitted for signal analysis applications:
  - In STFT with a certain time-frequency resolution, we are essentially incapable to notice both high and low frequency events; however in WT these restrictions do not exist.
  - Consequently, in WT, a wonderful capability is denoising the signal regarding a frequency band and certain time; on the other hand, filtering using FT does not allow different frequency bands at different times.

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Comparison of WT with STFT

- The usage of “Compactly Supported” mother wavelets gives us the opportunity to correlating signal with a basis function locally.
  - In STFT, basis functions are just multiplication of sinusoids by (usually) a Gaussian function. Rather in WT, the mother wavelets are generated based on diverse mathematical formulations or recursive solving of scaling functions

$$\psi(x) = C_p e^{-x^2} \qquad Gaussian\ Derivatives$$

$$\psi(x) = Ce^{-x^2}\cos(5x) \qquad Morlet$$

$$\psi(x) = \sqrt{f_b}\left(sinc\left(\frac{f_b x}{m}\right)\right)^m e^{2i\pi f_c x} \qquad Complex\ frequency\ B - Spline$$

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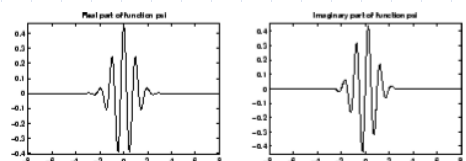
Comparison of WT with STFT

- Another advantageous characteristic of wavelets over STFT is the possibility of choosing "Complex" basis functions rather than real ones as we did in STFT.
  - This unique capability enables us to estimate complex mode shapes in the case of non-classical damping.
  - By choosing complex mother wavelets like "Complex Shannon", "Complex Morlet" we can have both phase and magnitude or real and imaginary parts



Comparison of WT with STFT

Fourier Transform	Wavelet Transform
Stationary signals	Non-stationary or transient signals
Frequency Information only, time/space information is lost	Joint time and frequency information
Single basis function	Many basis functions
High computational costs	Low computational costs
Few analysis procedures	Numerous analysis structures (more customization)

Continuous Wavelet Transform

$$W(a,b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} x(t)\psi^*\left(\frac{t-b}{a}\right) dt$$

Wavelet coefficient: W; Scale: a; Shift: b; Mother wavelet: ψ^*

where $\psi^*(.)$ denotes the complex conjugate of mother wavelet $\psi(t)$, and $\psi\left(\frac{t-b}{a}\right)$ (basis wavelet) is the translated and dilated (stretched) version of mother wavelet.

Continuous Wavelet Transform

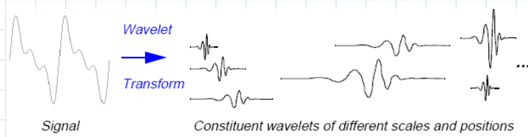
- The term $\frac{1}{\sqrt{a}}$ has been used in order to normalize the energy of this time-scale representation.
- The usage of compactly supported mother wavelets gives us the integral over a finite length rather than minus infinity to plus infinity as we have in FT.
- Use of mother wavelets with higher vanishing moment will increase precision of signal approximation
- The k^{th} moment of a wavelet is defined as:

$$m_k = \int t^k \psi(t) dt$$

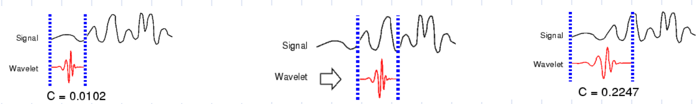
- If $m_0 = m_1 = m_2 = \cdots = m_{p-1} = 0$, we say $\psi(t)$ has p vanishing moments

Continuous Wavelet Transform

- Scalogram of wavelet transform could be produced by correlating the signal with wavelets at different scales and positions



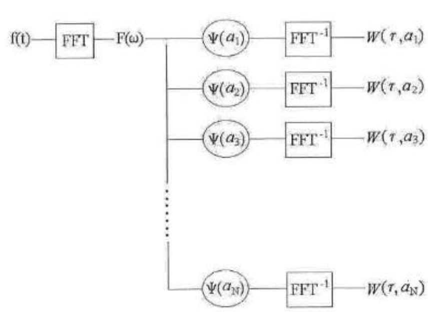
- Take a wavelet, compare it to a section at the start of the original signal and calculate correlation coefficient. Then shift the wavelet to the right to cover whole of the signal
- Then move to another scale and perform same procedure and finally construct time-scale map



Continuous Wavelet Transform

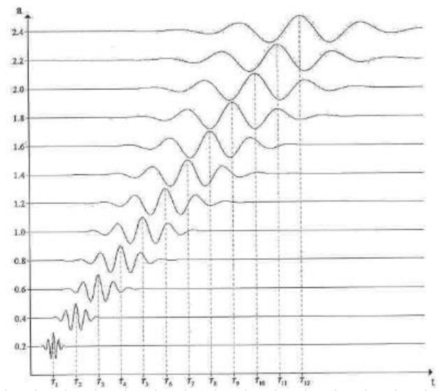
- WT also could be computed in frequency domain as presented by Parseval's theorem:

$$W_f(\tau, a) = \frac{\sqrt{a}}{2\pi} \int_{-\infty}^{\infty} F(\omega)\Psi^*(a\omega)e^{i\omega\tau} d\omega$$



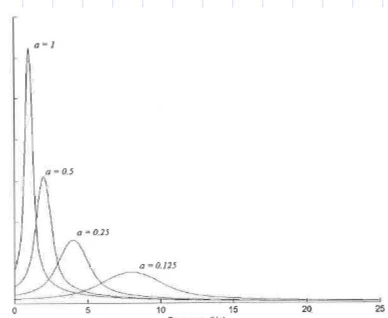
Continuous Wavelet Transform

- Effect of different scales and positions is illustrated and it is apparent that higher scales is corresponding to more stretched wavelets



Continuous Wavelet Transform

- Thus, WT can be seen as a bank of filters. Variation of scale parameter generates a set of filters in the frequency domain.
- As scale value decreases the filter reduces its amplitude and increases its bandwidth



Continuous Wavelet Transform

- Different mother wavelets might affect the results either in time- or scale- domain due to differences in both domains as in seen for Morlet and Mexican Hat wavelets

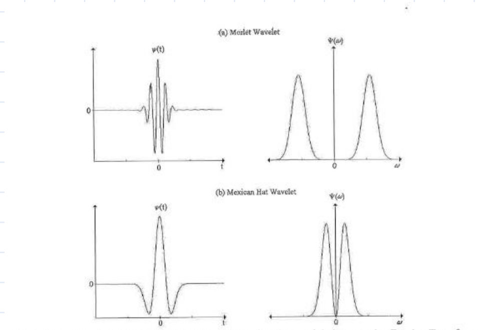


Figure 4.1: Different types of analytical wavelet basis functions and their respective Fourier Transform: (a) Morlet Wavelet, (b) Mexican Hat.

Continuous Wavelet Transform

- This transform is applicable for functions which are square-integrable denoted by $L^2(R)$ and define as:

$$\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$$

- Radius of time-frequency windows are as follows:

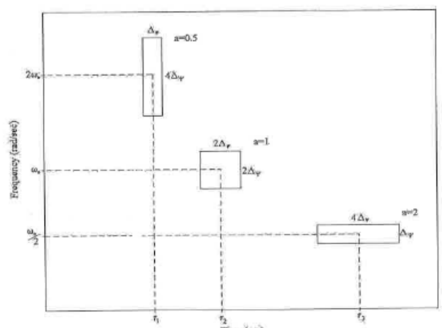
$$\Delta t_{\psi} = \sqrt{\frac{\int_{-\infty}^{\infty} (t - t_0)^2 |\psi(t)|^2 dt}{\int_{-\infty}^{\infty} |\psi(t)|^2 dt}}$$

$$\Delta \omega_{\Psi} = \sqrt{\frac{\int_{-\infty}^{\infty} (\omega - \omega_0)^2 |\Psi(\omega)|^2 d\omega}{\int_{-\infty}^{\infty} |\Psi(\omega)|^2 d\omega}}$$

Continuous Wavelet Transform

- Based on "Heisenberg Uncertainty Principle" the area of time-frequency is constant, such that:

$$\Delta t_{\psi(a)} \Delta\omega_{\Psi(a)} = |a| \Delta t_{\psi} \frac{\Delta\omega_{\Psi}}{|a|} = \Delta t_{\psi} \Delta\omega_{\Psi} \geq \frac{1}{2}$$



Discrete Wavelet Transform

- A Discrete Wavelet Transform (DWT) is any wavelet transform for which the wavelets are discretely sampled.
- As with other wavelet transforms, a key advantage it has over Fourier transforms is temporal resolution: it captures both frequency and location information (location in time).
- **Approximations** are High-scale, low-frequency components of the signal
- **Details** are low-scale, high-frequency components

Discrete Wavelet Transform

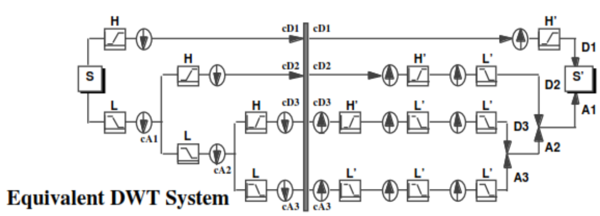
- Note that scales and positions are chosen dyadically in order to reduce the computation costs.
- In DWT, basis wavelet, scale and translation are:

$$\psi_{jk}(x) = 2^{\frac{j}{2}}\psi(2^j t - k) \qquad j,k \in \mathbb{Z}$$
$$a = 2^{-j}$$
$$b = k2^{-j}$$

- In this context, there exist a scaling function which behaves as low-pass filter and a wavelet function as high-pass filter.
- DWT has very complex mathematical background which is related to functional analysis and subspaces; for a more thorough description refer to books and other references in this field.

Discrete Wavelet Transform

- A simple 3-level DWT has shown in the following



Equivalent DWT System

- Frequency allocation in DWT is also has shown

$$S = A_1 + D_1 = A_2 + D_2 + D_1 = A_3 + D_3 + D_2 + D_1$$

Applications

- Scale-domain aspects:
  - **Biology** for cell membrane recognition
  - **Metallurgy** for the characterization of rough surfaces
  - **Finance** for detecting the properties of quick variation of values
  - In **Internet traffic** description, for designing the services size
- Time-domain aspects:
  - Rupture and **edges detection**
  - Study of **short-time phenomena** as transient processes

Applications

- Some other applications are:
  - Chaos Dynamics, **Intermittence** in physics
  - Partial Differential Equation solving e.g. buckley-Leverett equation
  - Compression of **fingerprints**
  - Quantum Mechanics, Turbulence
  - **Nondestructive control** quality processes
  - EEG, heart-rate, blood pressure, brain rhythms, DNA
  - **SAR** (Synthetic Aperture Radar) imagery, Astrophysics
  - **Oceanography**, earth studies, Seismology, Climatology
  - Molecular Dynamics
  - Identification of hydrocarbon and source rock
  - Water distribution systems, forecasting traffic volume

Applications

- However, WT has some very diverse applications in **engineering** such as:
  - "System Identification" for civil/mechanical/electrical engineering applications as an **output-only** approach
  - Particularly, in "Structural Vibration" for modal **damping** and **natural frequency** identification
  - In "Structural Health Monitoring" in order to detect the existence, location and severity of **damages**
  - In systems with **time-varying** or **nonlinear** behavior; in contrast to FT, WT is capable to monitor dynamical behavior of system
  - For computation of "**Impulse Response Functions**" and time-varying "**Frequency Response Functions**"

Applications

- For a **large number of matrix multiplication algorithms** in order to reduce substantially the time of computation; particularly, in cases such as determining "**Markov Parameters**" in the context of "Time-Domain System Identification" which require recursive multiplication of matrices, WT is beneficent in quicker multiplications
- In order to estimate physical parameters such **stiffness, damping and stiffness matrices** by having the whole mass of the structure
- Computation of time-varying **Holder exponent** which is proposed as a damage-sensitive feature by applying wavelet transform on both sides of Holder nonequality

Applications

- Wavelet transform has merits of less-model dependence, sensitivity to local damage, robustness to moderate noise, computational efficiency, and feasibility for on-line implementation.
- Wavelet transform has great potentials to be used in multi-level structural health monitoring for structures to detect, locate, and assess structural damage as well as to make a maintenance decision in condition-based maintenance procedure.

Applications

- Wavelet transform has great potentials for structural reliability analysis structures in Monte Carlo simulation, adaptive Bayesian reliability assessment, and life prediction.
- Wavelet-based sampling techniques have been widely used for random vibration analysis.
- Wavelet is capable for performing adaptive Bayesian system identification.
- Adaptive reliability assessment of critical structural members and prediction of their remaining life can also be done by WT.

Applications

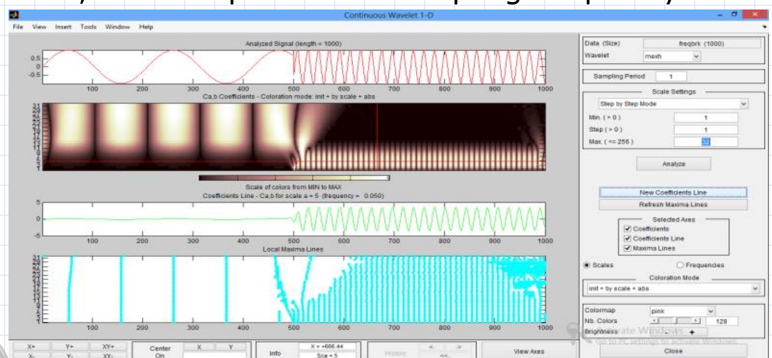
- In active vibration control, for time varying **pole assignment**
- Developing **simulated stochastic ground motion** model compatible with design spectrum with time and frequency nonstationarity using decomposing capabilities of this transform
- Estimation of **seismic response** of structures in linear/nonlinear systems using capability of WT for transforming differential equation of motion to algebraic equation of motion either for SDF/MDF systems
- "Mother Wavelets" could be used as "**Shape Functions**" in the field of "Finite Element Analysis"

MATLAB Toolbox

- Despite all the difficulties encountered with the "Wavelet Transform", we have a very handy tool for this transform: **Wavelet Toolbox** (`wavemenu`)

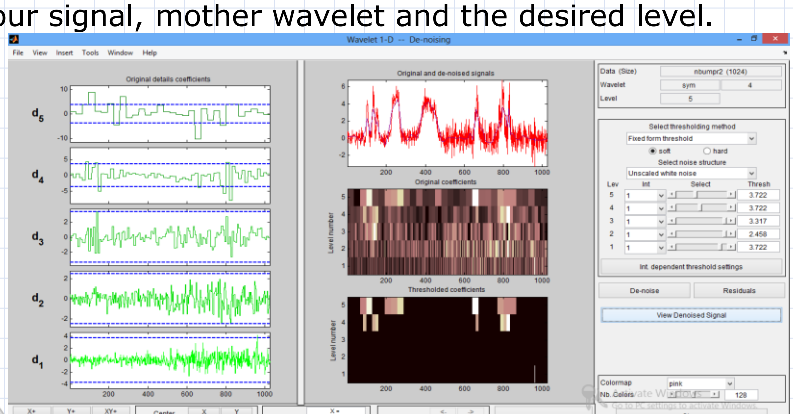
MATLAB Toolbox

- In order to perform a “Continuous Wavelet Transform” using the “Graphical Interface”, we can simply select our signal, mother wavelet, starting and ending scales, scale step and also sampling frequency



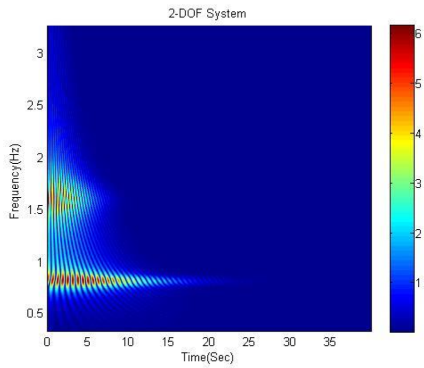
MATLAB Toolbox

- In order to perform a “Discrete Wavelet Transform” using the “Graphical Interface”, we can simply select our signal, mother wavelet and the desired level.



Practical Implementation in Structural Dynamics

- An output-only approach for "Modal Identification" as follows:



colors denote magnitude of wavelet transform

Practical Implementation in Structural Dynamics

- Obviously, two frequencies can be identified
- Based on frequencies which are extracted, it is possible to estimate modal damping ratios and real/complex mode shapes as follows:

$$\ln|W_\psi(a_0,b)| = -\xi\omega_n b + \ln(\frac{\sqrt{a_0}}{2}B|\psi^*(a_0\omega_d)|$$

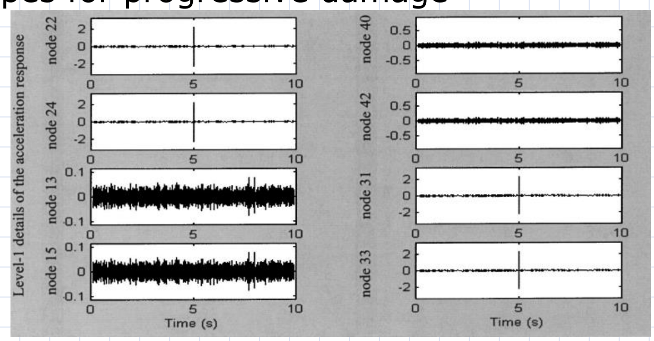
$$r_{k,j} = \frac{\sum_{l=1}^{N} Re\left[W_\psi^{x_k}(a_j,b_l)\right] Re\left[W_\psi^{x_{ref}}(a_j,b_l)\right]}{\sum_{l=1}^{N} Re^2\left[W_\psi^{x_{ref}}(a_j,b_l)\right]}$$

$$s_{k,j} = \frac{\sum_{l=1}^{N} Re\left[W_\psi^{x_k}(a_j,b_l)\right] Re\left[W_\psi^{x_{ref}}(a_j,b_l)\right]}{\sum_{l=1}^{N} Im^2\left[W_\psi^{x_{ref}}(a_j,b_l)\right]}$$

$r_{k,j}$, $s_{k,j}$ are real and imaginary parts of mode shape

Practical Implementation in Structural Dynamics

- Wavelet tools can be used to locate damage regions based on either spatial distribution of **spikes** for sudden damage or change in mode shapes for progressive damage



Practical Implementation in Structural Dynamics

- As it is seen, in DWT for nodes 22, 24, 31, 33 there is a spike at 5th second of the signal
- Each node having spike in the low levels of details means that the node is related to a high-frequency event
- High-frequency events could be an impulse or damage or sudden change in stiffness
- Another useful but difficult application of WT is the possibility of time-dependent denoising of non-stationary signals

References for Further Reading

- Conceptual Wavelets in Digital Signal Processing (D. L. Fugal)
- A Wavelet Tour of Signal Processing (S. Mallat)
- System Identification: Time-Frequency Methods (S. Nagarajaiah)
- A Practical Guide to Wavelet Analysis (C. Torrence and G. P. Compo)
- Introduction to Wavelets in Engineering (J. R. Williams and K. Amaratunga)
- Wavelet Transform for System Identification in Civil Engineering (T. Kijewski and A. Kareem)
- Output only Modal Identification and Structural Damage Detection using Time Frequency & Wavelet Techniques (S. Nagarajaiah and B. Basu)

References for Further Reading

- Online Identification of Linear Time-varying of Structural Systems by Wavelet Analysis (B. Basu, S. Nagarajaiah and A. Chakraborty)
- Time-frequency and Time-scale Analysis for SHM (W. J. Staszewski and A. N. Robertson)