An Introduction to Short-Time Fourier Transform (STFT)

Advanced Structural Dynamics

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Scope and Goals

- To expand the capabilities of Fourier transform for time-varying signals
- In addition to showing the frequency content of the signals, it is also desirable to have an idea of when each frequency content is dominant
- Describing complicated signals with fewer number of frequencies

Fourier Transform

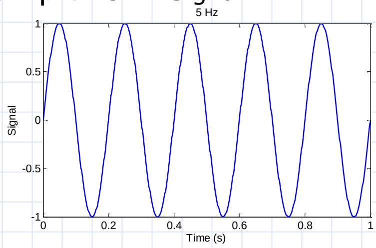
- Fourier analysis expands signals or functions in terms of sinusoids (or complex exponentials).
- It reveals all frequency components present in a signal.

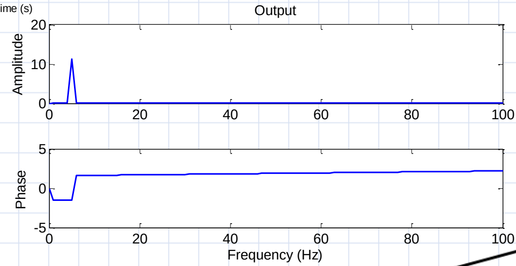
$$p_n = \sum_{j=0}^{N-1} P_j e^{i(j\omega_0 t_n)} = \sum_{j=0}^{N-1} P_j e^{i(2\pi nj/N)}$$ **Inverse DFT**

$$P_j = \frac{1}{T_0} \sum_{n=0}^{N-1} p_n e^{-i(j\omega_0 t_n)} \Delta t = \frac{1}{N} \sum_{n=0}^{N-1} p_n e^{-i(2\pi nj/N)}$$ **DFT**

Fourier Transform

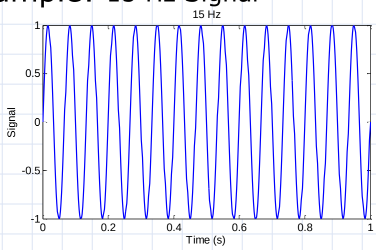
- Example: 5 Hz Signal

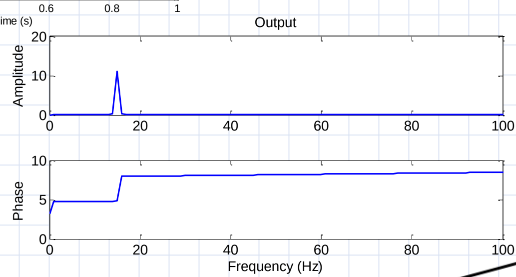




Fourier Transform

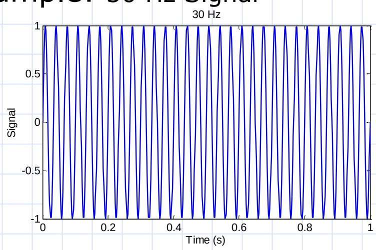
- Example: 15 Hz Signal

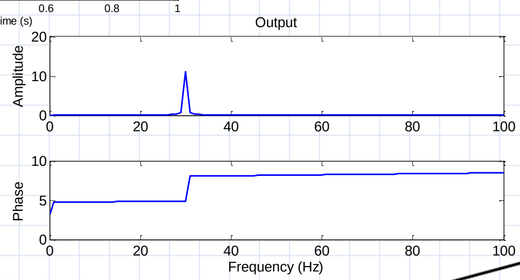




Fourier Transform

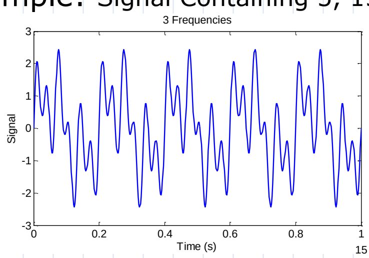
- Example: 30 Hz Signal

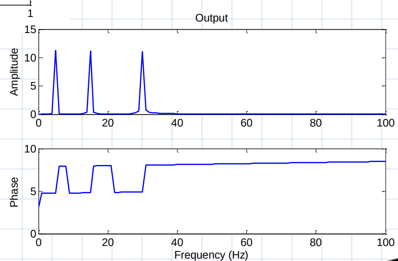




Fourier Transform

- Example: Signal Containing 5, 15 and 30 Hz Components





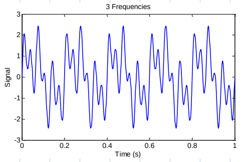
Fourier Transform - Limitations

- Cannot provide information about both time and frequency – i.e. cannot provide simultaneous time and frequency localization
  - → Not suitable for analyzing time-variant, non-stationary signals
    - As an example, it cannot be used to analyze the signals obtained from nonlinear or damaged structures, since they may contain varying frequency content at different times

Fourier Transform - Limitations

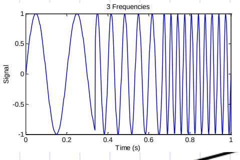
- Signals: Stationary vs Non-stationary
  - Stationary signals have properties (e.g frequency content) that does not change over time

Sum of 3 signals, present at all times



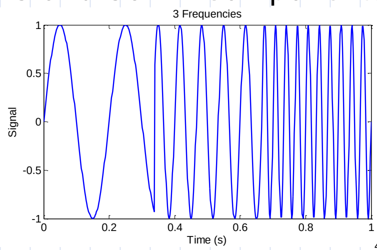
  - Non-stationary signals may have different frequencies at different times

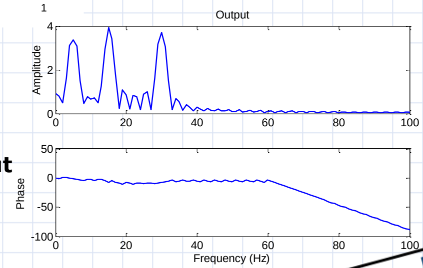
Combination of 3 signals occurring at different times



Fourier Transform - Limitations

- 5, 15 and 30 Hz Components at Different Times





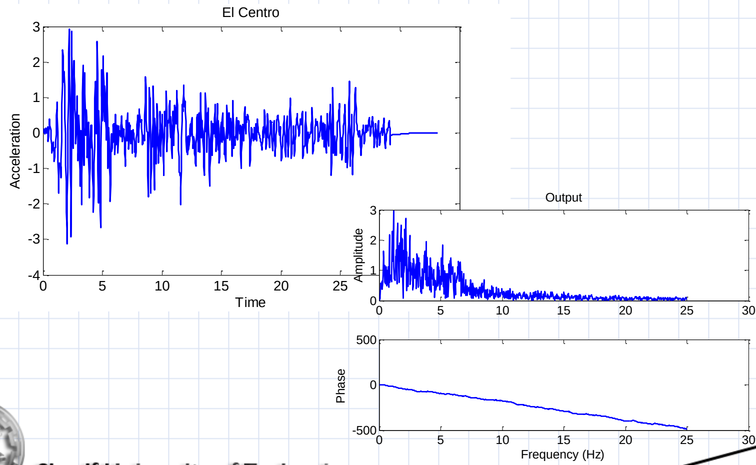
Excellent frequency localization (shows all existing frequencies), but no time localization

Fourier Transform - Limitations

- Not appropriate for representing discontinuities or sharp corners
  - → Requires a large number of Fourier components to represent discontinuities).

Fourier Transform - Limitations

- El Centro Earthquake Record and its Fourier transform



Fourier Transform - Limitations

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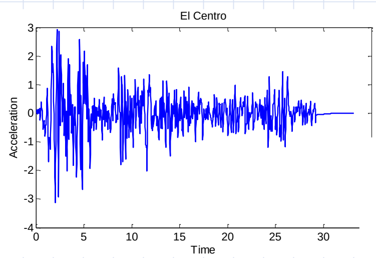
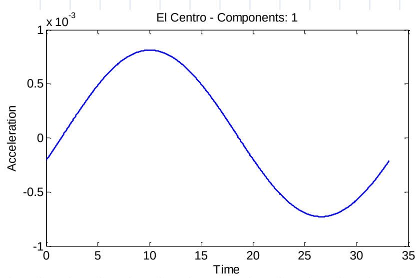


Included Signals: 1



Fourier Transform - Limitations

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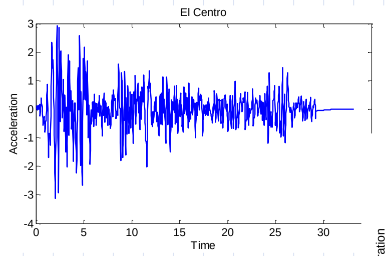
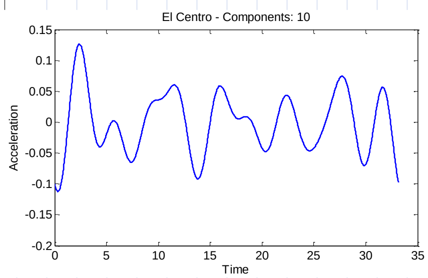


Included Signals: 10



Fourier Transform - Limitations

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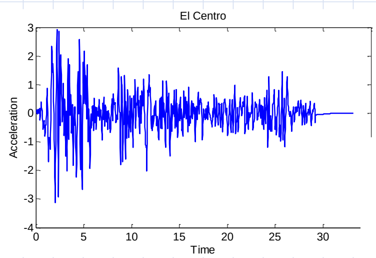
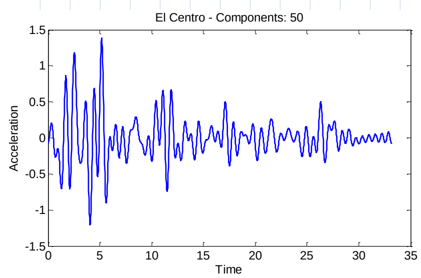
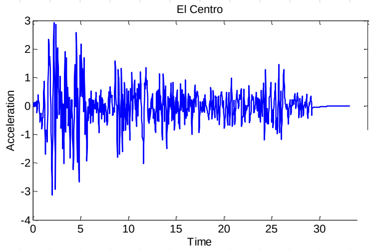


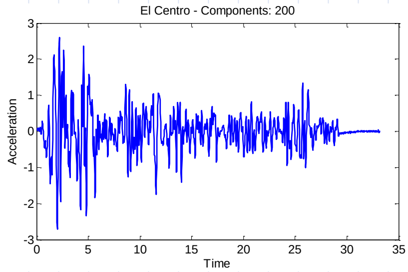
Included Signals: 50



Fourier Transform - Limitations

- El Centro Earthquake Record and its Fourier transform

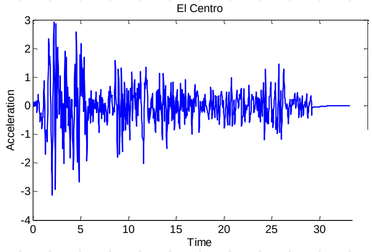


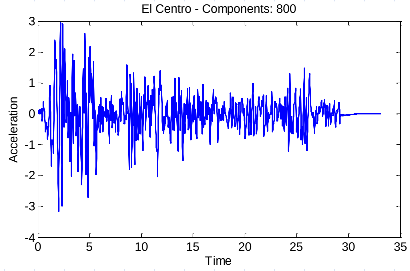


Included Signals: 200

Fourier Transform - Limitations

- El Centro Earthquake Record and its Fourier transform





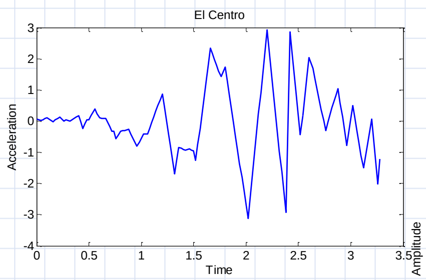
Included Signals: 800

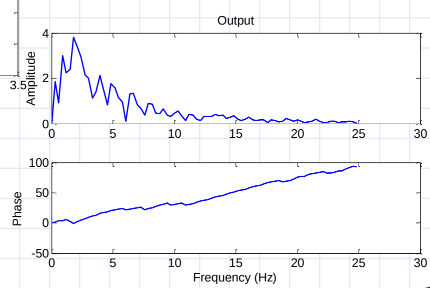
Short-Time Fourier Transform

- Basic Concept:
  - Break up the signal in time domain to a number of signals of shorter duration, then transform each signal to frequency domain
    - Requires fewer number of harmonics to regenerate the signal chunks
    - Helps determine the time interval in which certain frequencies occur

Short-Time Fourier Transform

- Consider the first 3 seconds of the El Centro Record





Short-Time Fourier Transform

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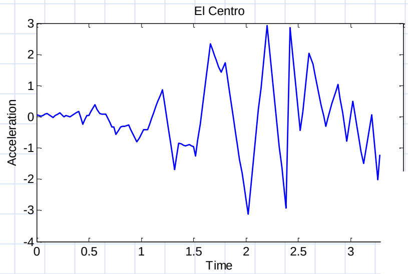
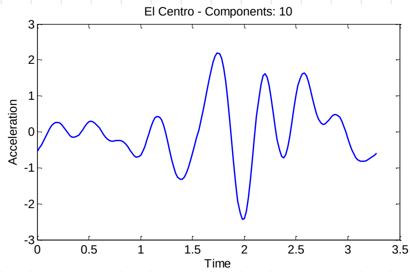


Included Signals: 10



Short-Time Fourier Transform

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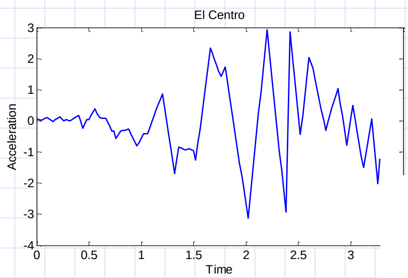
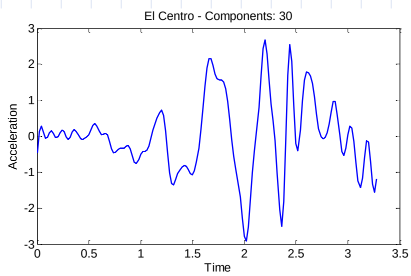
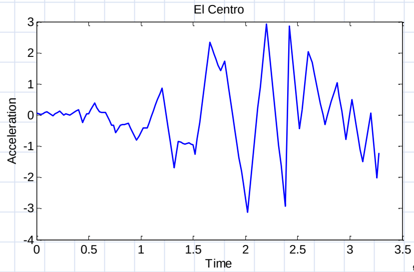


Included Signals: 30

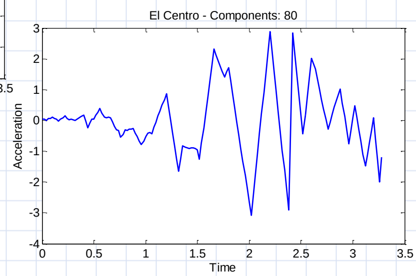


Short-Time Fourier Transform

- Consider the first 3 seconds of the El Centro Record

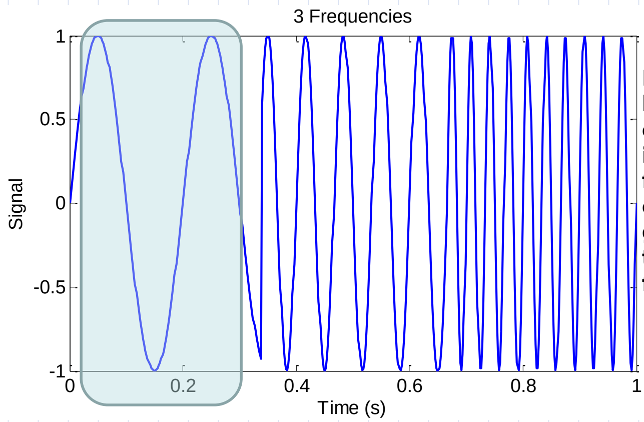


Included Signals: 80



Short-Time Fourier Transform

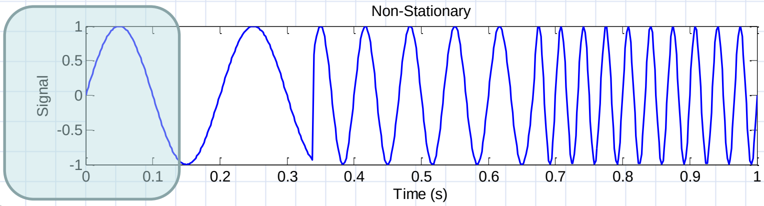
- Or consider the first chunk of non-stationary record



Knowing the location of the considered chunk in time and its frequency content, we can get an idea of the time each frequency exists.

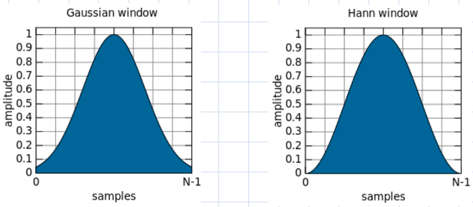
Short-Time Fourier Transform

- Need a ***local*** analysis scheme for a time-frequency representation (TFR).
- Windowed F.T. or Short Time F.T. (STFT)
  - Segmenting the signal into narrow time intervals (i.e., narrow enough to be considered stationary).
  - Take the Fourier transform of each segment.



Short-Time Fourier Transform

- Steps:
  - Choose a window function of finite length
    - A window function is a function that is multiplied by the signal to keep a certain portion of it



  - Place the window on top of the signal at t=0
  - Truncate the signal using this window

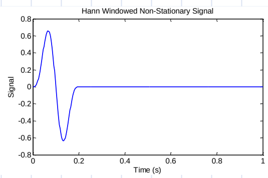
Short-Time Fourier Transform

- Steps (continued)
  - Compute the FT of the truncated signal, save results.
    - For each time location where the window is centered, we obtain a different FT
      - Each FT provides the spectral information of a separate time-slice of the signal, providing simultaneous time and frequency information
  - Incrementally slide the window to the right
  - Repeat until window reaches the end of the signal

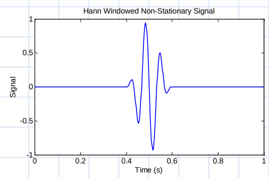
Short-Time Fourier Transform

- What is being Fourier-transformed:

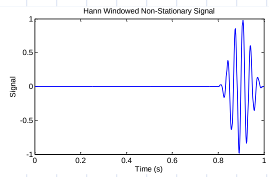
Window Length: 0.2s



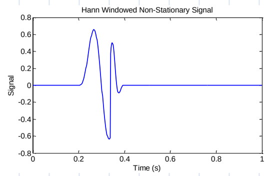
$\tau = 0.1\text{s}$



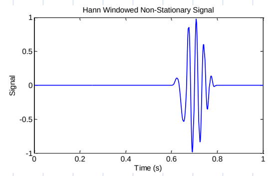
$\tau = 0.5\text{s}$



$\tau = 0.9\text{s}$



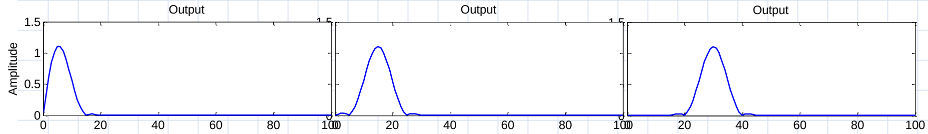
$\tau = 0.3\text{s}$



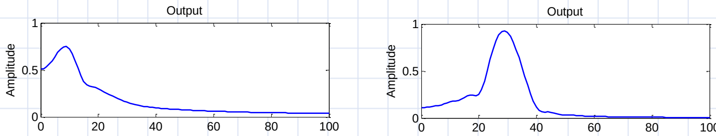
$\tau = 0.7\text{s}$

Short-Time Fourier Transform

- Fourier transforms of windowed signal:



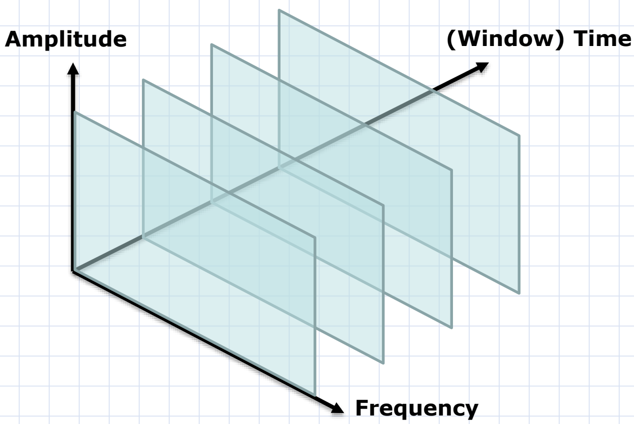
$\tau = 0.1\text{s}$ $\tau = 0.5\text{s}$ $\tau = 0.9\text{s}$



$\tau = 0.3\text{s}$ $\tau = 0.7\text{s}$

Short-Time Fourier Transform

- Now we can plot the spectra next to each other to generate a ***surface***:



Short-Time Fourier Transform

- Formulation:
  - Continuous STFT

$$\mathrm{STFT}\{x(t)\}(\tau,\omega) \equiv X(\tau,\omega) = \int_{-\infty}^{\infty} x(t)w(t-\tau)e^{-i\omega t}dt$$

Often phase unwrapping is employed along either or both the time axis, τ, and frequency axis, ω, to suppress any jump discontinuity of the phase result of the STFT

$x(t)$	Time-domain signal to be transformed
τ	Time (slow time; lower resolution than t)
ω	Frequency
$w(t)$	Window function, commonly a Hann window or Gaussian window bell centered around zero
$X(\tau,\omega)$	A complex function representing the phase and magnitude of the signal over time and frequency (this is essentially the Fourier Transform of $x(t)w(t-\tau)$)

Short-Time Fourier Transform

- Formulation:
  - Discrete STFT

Time discretized, frequency continuous; but if FFT is used, they both will be discrete.

$$\mathrm{STFT}\{x_n\}(m,\omega) \equiv X(m,\omega) = \sum_{n=-\infty}^{\infty} x_n w_{n-m} e^{-i\omega t_n}$$

x_n	Sequence of discretized time-domain signal to be transformed
m	Time index
ω	Frequency
w_n	Sequence of discretized window function
$X(m,\omega)$	STFT of the time-domain sequence

Short-Time Fourier Transform

- Inverse STFT
  - The original signal can be recovered from the transform by the Inverse STFT. The most widely accepted way of inverting the STFT is by using the overlap-add (OLA) method, which also allows for modifications to the STFT complex spectrum.
- Calculating the Inverse STFT:
  - First, it is required that the window function must be scaled such that the area underneath the window function is unity:

$$\int_{-\infty}^{\infty} w(\tau)\,d\tau = 1$$

Short-Time Fourier Transform

- Calculating the Inverse STFT (Continued):
  - It follows that:

$$\int_{-\infty}^{\infty} w(t-\tau)\,d\tau = 1 \quad \forall\, t$$

  - Hence:

$$x(t) = x(t)\int_{-\infty}^{\infty} w(t-\tau)\,d\tau = \int_{-\infty}^{\infty} x(t)w(t-\tau)\,d\tau$$

  - The continuous Fourier transform is:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}\,dt$$

  - Substituting from above:

$$X(\omega) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(t)w(t-\tau)\,d\tau\right] e^{-i\omega t}\,dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)w(t-\tau)\,e^{-i\omega t}\,d\tau\,dt$$

Short-Time Fourier Transform

- Calculating the Inverse STFT (Continued):
  - Swap the integration order:

$$X(\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)w(t-\tau)\, e^{-i\omega t}\, dt\, d\tau$$
$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(t)w(t-\tau)\, e^{-i\omega t}\, dt\right] d\tau$$
$$= \int_{-\infty}^{\infty} X(\tau,\omega)\, d\tau$$

  - Since the inverse Fourier transform is:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{+i\omega t}\, d\omega$$

  - Then the time-domain signal can be recovered:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(\tau,\omega) e^{+i\omega t}\, d\tau\, d\omega$$

Short-Time Fourier Transform

- Calculating the Inverse STFT (Continued):
  - Or:

$$x(t) = \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\tau,\omega) e^{+i\omega t}\, d\omega\right] d\tau$$

  - Alternatively, one can obtain windowed *grain* or *wavelet* of *x*(*t*) as:

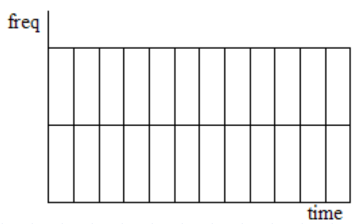
$$x(t)w(t-\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\tau,\omega) e^{+i\omega t}\, d\omega$$

Short-Time Fourier Transform

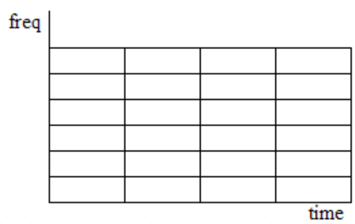
- Time and Frequency Resolution:
  - The STFT has a fixed resolution
    - The width of the windowing function relates to how the signal is represented
    - It determines whether there is good frequency resolution (frequency components close together can be separated) or good time resolution (the time at which frequencies change).
    - A wide window (wideband transform) gives better frequency resolution but poor time resolution.
    - A narrower window (narrowband transform) gives good time resolution but poor frequency resolution.

Short-Time Fourier Transform

- Time and Frequency Resolution (Continued):
  - That is, we cannot have good resolutions in both time and frequency.



Good time resolution, poor frequency resolution



Good frequency resolution, poor time resolution

Short-Time Fourier Transform

- Time and Frequency Resolution (Continued):
  - To explain this limitation, note that in Fourier transform:
    - To increase the frequency resolution of the window the frequency spacing of the coefficients (sequence in frequency domain) needs to be reduced.
      - Decreasing Nyquist (max) frequency (and keeping N constant) will cause the window size to increase — since there are now fewer samples per unit time.
      - The other alternative is to increase N, but this again causes the window size to increase.
      - So any attempt to increase the frequency resolution causes a larger window size and therefore a reduction in time resolution—and vice-versa.

Short-Time Fourier Transform

- Time and Frequency Resolution (Continued):
  - Best simultaneous resolution of both is reached with a Gaussian window function. This STFT with some modifications for multi-resolution becomes the Morlet wavelet transform.
  - Window should be narrow enough to make sure that the portion of the signal falling within the window is stationary.
  - Very narrow windows do not offer good localization in the frequency domain.

Short-Time Fourier Transform

- Time and Frequency Resolution (Continued):
  - **Windowing Function infinitely long**: $w(t) = 1$
    - STFT turns into FT, providing excellent frequency localization, but no time information.

$$X(\tau,\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt = X(\omega)$$

  - **Windowing Function infinitely short**: $w(t) = \delta(t)$
    - gives the time signal back, with a phase factor, providing excellent time localization but no frequency information.

$$X(\tau,\omega) = \int_{-\infty}^{\infty} x(t)\delta(t-\tau)e^{-i\omega t}dt = x(\tau)e^{-i\omega\tau}$$

Short-Time Fourier Transform

- Heisenberg (Uncertainty) Principle:

$$\Delta t \, \Delta f \geq \frac{1}{4\pi}$$

Time resolution: How well two spikes in time can be separated from each other in the transform domain.

Frequency resolution: How well two spectral components can be separated from each other in the transform domain.

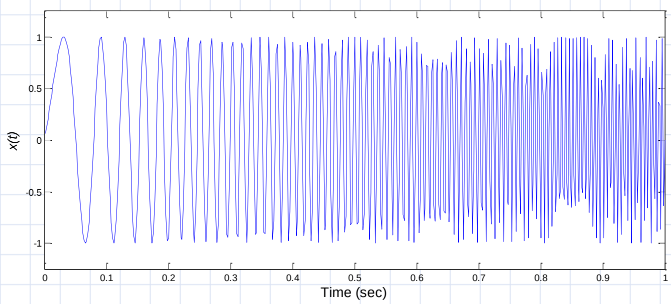
$\boldsymbol{\Delta t}$ ***and*** $\boldsymbol{\Delta f}$ ***cannot be made arbitrarily small*** !

Short-Time Fourier Transform

- Heisenberg (Uncertainty) Principle (Continued):
  - One cannot know the *exact* time-frequency representation of a signal.
  - We cannot precisely know at what time instance a frequency component is located.
  - We can only know what *interval of frequencies* are present in which *time intervals.*

Short-Time Fourier Transform

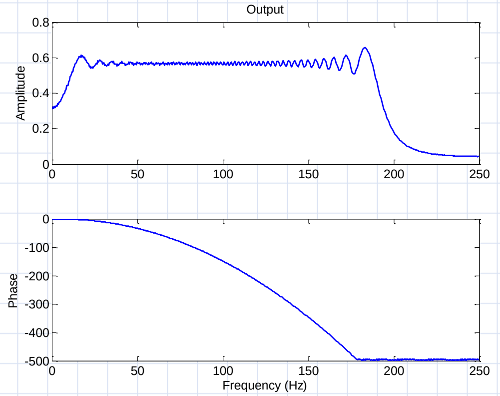
- Example:
  - Chirp Signal (linear frequency variation 10-200 Hz)



500 points spaced at 0.002 s, variable window sizes

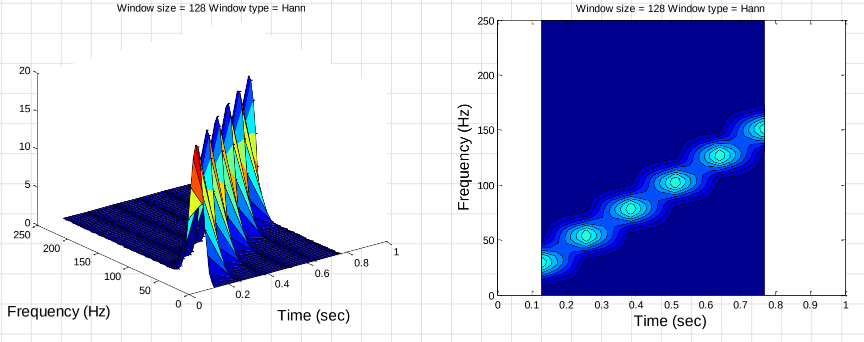
Short-Time Fourier Transform

- Example (Continued):
  - Chirp Signal – Ordinary Fourier Transform



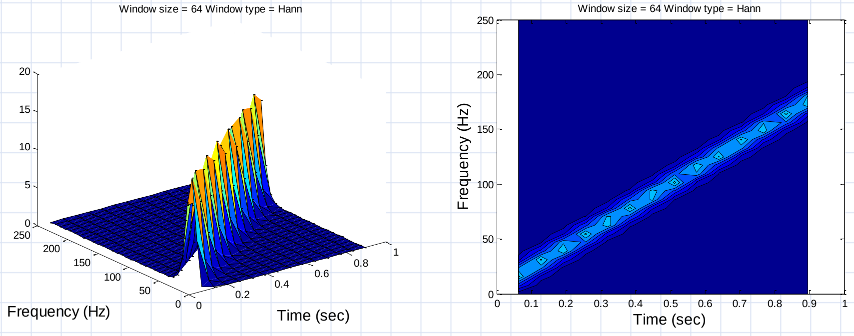
Short-Time Fourier Transform

- Example (Continued):
  - Chirp Signal – Window Size: 128 points



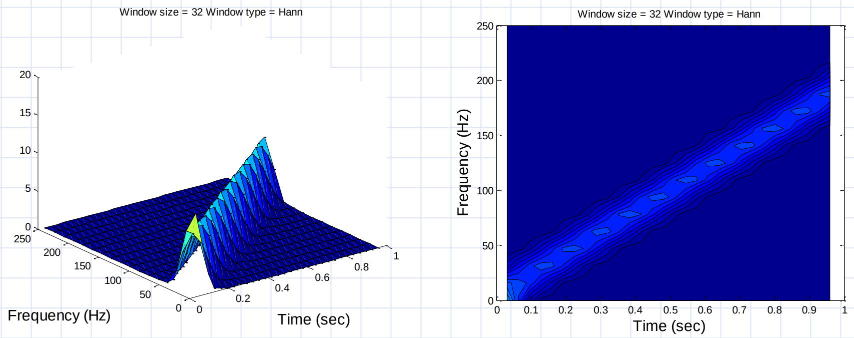
Short-Time Fourier Transform

- Example (Continued):
  - Chirp Signal – Window Size: 64 points



Short-Time Fourier Transform

- Example (Continued):
  - Chirp Signal – Window Size: 32 points



Short-Time Fourier Transform

- Use of MATLAB:
  - The MATLAB command to perform STFT is spectrogram:

    [B, f, t] = spectrogram(x, window, noverlap, nfft, fs)

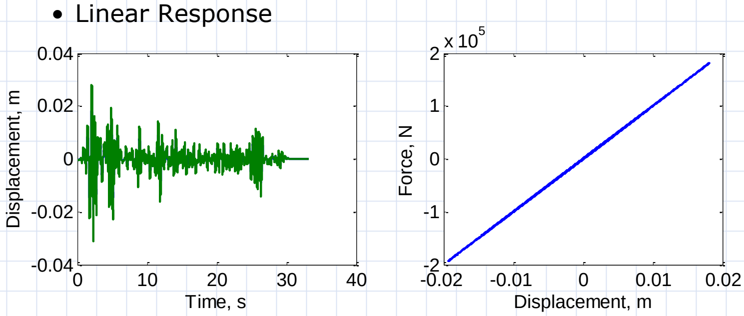
  - Spectrogram inputs:
    - x: input signal (signal to be transformed)
    - window function (discrete window at the same input rate)
    - window overlap (in terms of number of points)
    - sampling points for FFT (number of points used in FFT)
    - sampling frequency (rate of sampling for the input signal and the window function)
  - Spectrogram outputs: Transformed signal, frequencies and times.

Short-Time Fourier Transform

- Applications:
  - Signal processing of any non-stationary signal (audio signals, earthquake excitations, structural responses to ambient vibrations, …)
  - In structural dynamics, STFT can be used to:
    - Determine the dominant modes of vibration (and their shapes and frequencies) at any time interval
    - Health monitoring and damage detection through the study of dominant frequencies
      - e.g. a reduction in frequency is generally indicative of damages leading to softer structures

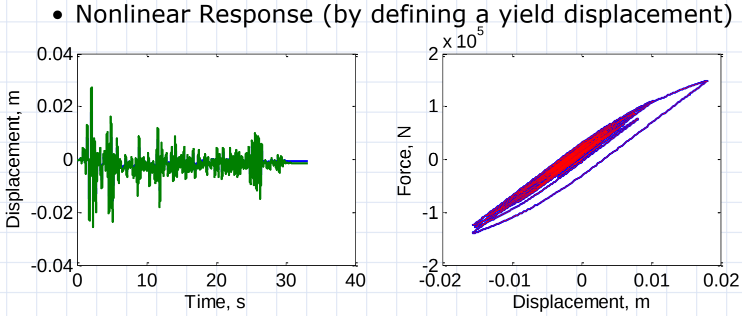
Short-Time Fourier Transform

- Applications:
  - Damage Detection Example:
    - 2-Story structure, natural frequencies 2.2 and 5.8 Hz
    - Response simulated to El Centro record
    - Linear Response



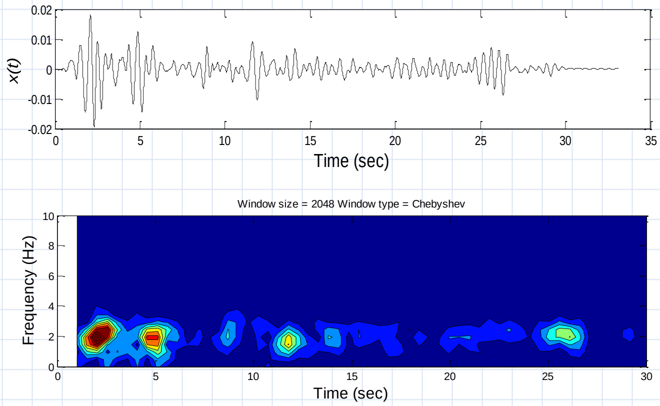
Short-Time Fourier Transform

- Applications:
  - Damage Detection Example:
    - 2-Story structure, natural frequencies 2.2 and 5.8 Hz
    - Response simulated to El Centro record
    - Nonlinear Response (by defining a yield displacement)



Short-Time Fourier Transform

- Applications:
  - Damage Detection Example: Linear Response



Short-Time Fourier Transform

- Applications:
  - Damage Detection Example: Nonlinear Response