Estimation and Stability over AWGN Channel in the Presence of Fading, Noisy Feedback Channel and Different Sample Rates

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Abstract

This paper is concerned with estimation and stability of control systems over communication links subject to limited capacity, power constraint, fading, noisy feedback, and different transmission rate rather than system sampling rate. A key issue addressed in this paper is that in the presence of noisy feedback associated with channel, which models transmission of finite number of bits over such links as is the case in most practical scenarios, the well-known eigenvalues rate condition is still a tight bound for stability. Based on an information theoretic analysis, necessary conditions are derived for stability of discrete-time linear control systems via the distant controller in the mean square sense. By construction of a specific coding scheme and by the design of a proper controller, the tight sufficient conditions for stability of control systems are also derived for linear discrete-time control systems over both Additive White Gaussian Noise (AWGN) and fading channels. This implies that the proposed coding scheme is efficient. The results are presented with the mismatch assumption between channel symbol and plant sample rates. As one key result of this paper, it is shown that if the channel symbol rate is less than the system sample rate, the control system is still stabilizable via increasing the transmission power.

Keywords: Networked control system, estimation, stability, feedback channel, capacity, mismatch factor.

1. Introduction

1.1. Motivation and Background

Networked Control Systems (NCSs) have attracted lots of interest in recent years. One characteristic of such distributed systems is that their components need to communicate with each other over communication networks that are subject to imperfections. This topic is also becoming of high interest as future generation of mobile communications, such as 5G, are explicitly intended to meet latency requirements for control applications [1]. For example, the tactile Internet which has been proposed within such context, refers to systems with low latency that allow real-time interactions with the environment through remote tactile control [2].

In NCSs, one important issue is how to estimate the system states in real-time at the distant controller and subsequently stabilize the system. The imperfect communication channel can be modeled based on the environment in which NCSs are deployed, leading to different practical system design paradigms [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19].

As the system output is continuous alphabet and the input and output of the Gaussian channels are also continuous alphabet, many works have studied the problems of controlling dynamic

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systems over AWGN channel, e.g., [5, 6, 7, 9, 11, 12, 13, 14, 15, 16, 17, 18, 19]. In the aforementioned papers, a number of issues, such as estimation, stability, and performance were addressed. However, in the case of discrete-time systems, all earlier works have assumed that the plant output sample rate and the channel input symbol rate are the same. Addressing estimation and stability with mismatch assumption between plant sample rate and channel symbol rate is important because in general, they can be different as it is considered in this paper. Also, the mismatch assumption between these two rates may be restrictive. For example, in [16], the stability condition for sampled continuous-time control systems is considered, where the bandlimited AWGN channels have been considered in the link from the plant to the controller and also from the controller to the plant. However, as the plant output sample rate and the channel input rate have been assumed to be the same in [16], the control system cannot be stabilized for some plant sample rates. Nevertheless, using our proposed coding scheme, the stability conditions for different sampling rates can be obtained in this paper.

In [19], it is shown that in the presence of AWGN channel with perfect feedback, the mean square stability of noisy discrete time linear systems is possible and the well-known eigenvalues rate condition (i.e., $C \geq \sum_{j=1;|\lambda_j(A)|>1} \ln|\lambda_j(A)|$, where C is the channel capacity in nats per time instant and $\lambda_j(A)$s are the eigenvalues of the linear system) is tight; while in the absence of this feedback link, the stability is impossible. This raises the fundamental question of what would happen if this feedback link is noisy? Addressing this question is important because noisy link models transmission of finite number of bits as is the case in most practical scenarios.

Furthermore, the fading channels have been the subject of interest in recent years as is the case in practical scenarios and studied in different models [4], [6], [8], [20], [21], [22], [23], [24]. In [20], the effect of fading has been modeled as variations in packet noise variance where the high variance noisy packets are dropped. Some articles have also modeled fading as a random multiplying factor without additive noise [4], [21], [22], [23]. [8] modeled fading as a digital channel with different Markov rate, and [6] considered multiplying flat fading AWGN broadband channel for continuous-time systems. In [24], a tight condition for stabilizing a noiseless dynamic systems over an AWGN fading channel is obtained.

1.2. Paper Contributions

The NCS considered in this paper is shown in Fig. 1, where the communication channel is assumed to be an AWGN channel with specific symbol rate. We also consider the same scenario with the AWGN channel replaced by a fading AWGN channel with perfect Channel Side Information (CSI) at receiver and transmitter sides. The basic NCS of Fig. 1 has been considered in many research papers, e.g., [3], [4], [6], [7], [19]. In this basic NCS block diagram, the communication from plant to the distant controller is subject to communication imperfections; while the communication from distant controller to the plant is perfect. This is the case, for example, in the tele-operation of micro autonomous vehicles, where the vehicle is supplied by limited capacity power supply and hence transmission from the vehicle to the remote base station where the controller is located is performed with low power and so it is subject to communication imperfections; while as the base station can be supplied with high power, the communication from distant controller to vehicle is almost perfect.

To address the aforementioned open problems, this paper first address the problem of estimation and stability with different sample rates; and it presents an extended version of the well-known eigenvalues rate condition for this case. All earlier works for discrete time systems are based on the assumption that there is only a single channel transmission between any two plant samples, which may not be the case in many practical scenarios. For example, in the tele-operation of autonomous road vehicles, the transmission link can be utilized at a higher rate to improve the control system performance. On the other hand, in the tele-operation of Autonomous Underwater

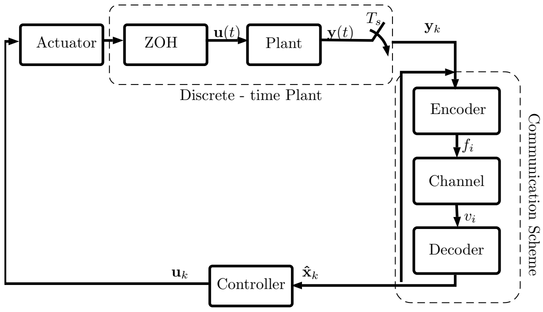

Figure 1: Control/communication system over AWGN channel

Vehicles (AUVs) that are controlled over a wireless acoustic channel with very limited bandwidth, transmission of a channel symbol may span over several plant samples. If the channel symbol rate is less than plant sample rate, previous schemes (e.g., [16]) leads to instability even for the case of high capacity channels, due to accumulative transmission delay. In the reverse case, if set the goal to transmit every plant sample once or a number of times (using repeated codes) through the channel, a tight bound on the stability condition cannot be achieved. However, by using the scheme proposed in this paper, the optimal design is achieved.

Secondly, this paper addresses the problem of estimation and stability of noisy linear systems over AWGN channel with noisy feedback link and noisy link from the distant controller to on-board plant actuator; and it shows that the eigenvalues rate condition is still a tight bound for stability for this case. Earlier works are based on perfect feedback link between transmitter and receiver and also perfect link from distant controller to actuator in order to obtain sufficient conditions for control objectives [6],[7],[12],[19], [25]. In practice, irrespective of whether transmission is performed over a noisy link or through a finite number of quantization bits, some level of uncertainty is introduced as the signal passes through the link.To the best of our knowledge, for the first time, the tight bound on the mean square stability of system over AWGN channel under such scenarios is achieved in this paper. It should be noted that some articles such as [26] consider noisy channels in both forward and reverse paths, however, in the presence of noiseless feedback channels. Furthermore, there are several works which consider imperfect feedback channel only in the case of packet erasure channel, e.g., [27, 28]. Also, the authors in [29] compared the solution of LQG problem in the cases of noiseless and noisy feedback channels.

Finally, in this paper, we also address a noiseless dynamic system over AWGN fading channels with noiseless output feedback, where the CSI is available instantaneously at the receiver but with delay at the transmitter side, while the mismatch assumption between channel symbol and plant output rates are also taken into account at the same time. The similar channel model is also considered in [24] for which a tight necessary and sufficient condition on the stability is obtain in the mean square sense for $m = 1$. In this paper, by choosing different criteria for the stability in the case of AWGN fading channel, we propose a different structure to stabilize dynamic systems under condition $mC \geq \sum_{j=1;|\lambda_j(A)|>1} \ln |\lambda_j(A)|$ which is tight and less restrictive than the obtained condition in [24] in the special case of $m = 1$.

1.3. Paper Organization

In the next part, we first present the necessary and sufficient conditions for discrete-time systems over AWGN and fading channels using a new coding/decoding scheme which utilizes different sample rates. Then, we address a real scenario in which there are some level of noise in communication feedback and controller-actuator links. Extension to the multi-dimensional systems is also addressed.

The paper is organized as follows: In Section II, the problem formulation is given. In Section

III, we present necessary and sufficient conditions for different scenarios of scalar discrete-time systems. Extension of the results to the vector case is presented in Section IV. Finally, Section V concludes the paper.

2. Problem formulation

2.1. Notations

We use the following notations throughout this paper. $|.|$ denotes the absolute value and $\|.\|$ the Euclidean norm. $\ln(.)$ is the natural logarithm, $E[.]$ denotes the expected value function, $(.)^{tr}$ denotes the transpose of matrix, $(.)^{+}$ denotes pseudo-inverse of matrix, and $det(.)$ is the function that computes the determinant of a square matrix. $trace(.)$ denotes the sum of the diagonal elements of matrix, $\lfloor . \rfloor$ maps the real number to the smallest next integer, $mod(a,b)$ finds the remainder of division of number a by b, z^k describes the sequence $(z_0, z_1, \ldots, z_k)$, $\mathbf{z}^{(j)}$ denotes the jth element of the vector $\mathbf{z}$, $\{A\}_{ij}$ represents the ijth element of the matrix A where the first subscript is the row number and the second is the column number and $\hat{\mathbf{x}}_{k|n}$ is the estimation of $\mathbf{x}_k$ given the observation $\mathbf{v}_0, \mathbf{v}_1 \ldots, \mathbf{v}_n$. $diag(\mathbf{v})$, I_n and $\underline{0}_n$, respectively denote, square diagonal matrix with the elements of vector $\mathbf{v}$ on the main diagonal, identity matrix of size n and zero vector with n elements. $N(m, Q)$ denotes Gaussian distribution with mean m and covariance matrix Q. $\exp(.)$ denotes the exponential function. $\lambda(A)$ denotes the eigenvalue of the matrix A. Finally, $\mathbb{R}$ and $\mathbb{W}$ denote the set of real numbers and non-negative integer numbers, respectively. As we consider mismatch factor, the number of generated plant samples is not necessarily equal to the number of transmitted channel symbols. Therefore, we illustrate the index of channel symbol with i and the index of the plant states with k.
In this paper, we use lower-case and lower-case boldface, respectively, for scalar and vector variables.

2.2. Preliminaries

Fig. 1 represents the system model considered in this paper. In what follows, we describe the input-output relation of each building block of the system.
Plant: discrete-time plant considered in this paper is described by the following linear, time-invariant and fully-observed system:

$$\begin{aligned} \mathbf{x}_{k+1} &= A\mathbf{x}_k + B\mathbf{u}_k + G\mathbf{w}_k, \\ \mathbf{y}_k &= \mathbf{x}_k, \end{aligned} \tag{1}$$

where $\mathbf{x}_k \in \mathbb{R}^d$ is the state, $\mathbf{u}_k \in \mathbb{R}^s$ is the control input, $\mathbf{w}_k \in \mathbb{R}^l$ is the process noise, $\mathbf{w}_k \sim N(\underline{0}_l, I)$, and $\mathbf{y}_k$ is the observation signal. It is assumed that (A, B) is a stabilizable pair and $\mathbf{x}_0$ is a random variable with bounded entropy.
Communication channel: We assume that there is a discrete-time AWGN fading channel between sensor and controller described by

$$v_i = \zeta_i f_i + n_i, \tag{2}$$

where $f_i \in \mathbb{R}$ denotes the channel input, $v_i \in \mathbb{R}$ the channel output, ζ_i is the channel state which is available at the receiver instantaneously and with one delay at transmitter, and $n_i \in \mathbb{R}$ the additive white Gaussian noise with zero mean and variance $N_0/2$ which models the uncertainties in this networked control system. We assume ζ_i is an independent identical distribution (i.i.d.) random process with the distribution:

$$P\{\zeta_i = \zeta_l\} = \gamma_l, \text{ for } i \in \mathbb{W},$$

where $l \in \{0, 1, \dots, s\}$, and $\gamma_l > 0$s are constant numbers, where $\sum_{l=1}^{s} \gamma_l = 1$. Assume that $\frac{1}{T_c}$ symbols per second are transmitted through the channel. If the CSI is available at the receiver instantaneously and with one delay at the transmitter, the capacity of this channel with output noiseless feedback under average power constraint $P_{av} \leq P$ is computed as $C = \frac{1}{2} E_{\zeta_i}(\ln(1 + \frac{\zeta_i^2 P}{N_0/2}))$ which is also the capacity of channel with no output feedback [30], where $E_{\zeta_i}(.)$ denotes the expected value function with respect to channel state ζ_i.
Note that, for a given input signal, the average power can be computed as $P_{av} = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} E[f_i^2]$ which is assumed to be less than P due to the average power constraint. If the channel states are deterministic and $\zeta_i = 1$ for every $i \in \mathbb{W}$, the channel is in fact a discrete-time AWGN channel, i.e.,

$$v_i = f_i + n_i. \tag{3}$$

Channel and process noise are assumed to be independent from each other and also from the initial state. The key parameter of interest in our analysis is the mismatch factor between plant sample rate and communication channel symbol rate, $m \triangleq \frac{T_s}{T_c}$. As the plant sampling rate and channel symbol rate are, in general, set independently, there is no reason for them to be equal. For example, in the case of $m = 2$, it is possible to transmit one additional symbol between two consecutive plant samples.
To compensate the effects of transmission noise, power constraint and fading, we need to use a proper encoder and decoder.
Encoder: We present two class of encoders. One maps $(\mathbf{x}_0, v^{i-1}, \zeta^{i-1}) \to f_i$ and the other maps $(\mathbf{x}_{\lfloor \frac{i}{m} \rfloor}, v^{i-1}, \zeta^{i-1}) \to f_i$ for $i = \{1, 2, \dots\}$. In this paper, in the case of no process noise, we can design the encoder of the first class since $\mathbf{x}_0$ can be obtained from $\mathbf{x}_{\lfloor \frac{i}{m} \rfloor}$ and vice versa.
Decoder: The decoder is the operator that maps $(\mathbf{u}^{k-1}, v^{\lfloor mk \rfloor}, \zeta^{\lfloor mk \rfloor}) \to \hat{\mathbf{x}}_{k|\lfloor mk \rfloor}$ at time instant kT_s for $k = \{0, 1, \dots\}$.
Controller: The controller generates the input of plant based on the value of $\hat{\mathbf{x}}_{k|\lfloor mk \rfloor} \in \mathbb{R}^d$. We use a linear controller, i.e. $\mathbf{u}_k = \mathcal{G}_k \hat{\mathbf{x}}_{k|\lfloor mk \rfloor}$, where the controller gain $\mathcal{G}_k$ should be designed to ensure stability as defined below.

Definition 1. *System (1) over the aforementioned communication link is bounded asymptotically (respectively, asymptotically) stabilizable in mean square sense if there exists an encoder, decoder and controller gain $\mathcal{G}_k$ such that $\lim_{k \to \infty} E[(\mathbf{x}_k)^{tr}(\mathbf{x}_k)] < \infty$ (respectively, $\lim_{k \to \infty} E[(\mathbf{x}_k)^{tr}(\mathbf{x}_k)] = 0$).*

Remark 1. *In the case of fading channel, the asymptotic stability is defined in the sense of $\lim_{k \to \infty} E[(\mathbf{x}_k)^{tr}(\mathbf{x}_k) | \zeta^{\lfloor mk \rfloor}] \stackrel{a.s}{=} 0$, where $\stackrel{a.s}{=}$ denotes the almost sure convergence.*

3. Discrete-time systems with mismatch factor

In this section, we present a necessary condition for stability of discrete-time invariant linear systems and propose two classes of encoder-decoder designs to obtain sufficient conditions. An important contribution in this part is that source and channel symbol rates are not restricted to be equal.

3.1. Necessary condition

In the following, we obtain a stability necessary condition for dynamic system (1) over AWGN fading channel (2) in terms of its capacity. For this channel model, the capacity is the same in the presence and absence of output noiseless feedback. Therefore, this theorem presents a necessary condition in the case of AWGN fading channel with perfect output feedback as well as without feedback or with noisy feedback.

It should be noted that for the case of AWGN channel, the condition $\lim_{k\to\infty} E[(\mathbf{x}_k)^{tr}(\mathbf{x}_k)|\zeta^{\lfloor mk \rfloor}] \overset{a.s}{<} \infty$ is equivalent to the condition $\lim_{k\to\infty} E[(\mathbf{x}_k)^{tr}(\mathbf{x}_k)] < \infty$. Hence, the following theorem which presents the stability necessary condition in the sense of $\lim_{k\to\infty} E[(\mathbf{x}_k)^{tr}(\mathbf{x}_k)|\zeta^{\lfloor mk \rfloor}] \overset{a.s}{<} \infty$ also implies the stability necessary condition in the mean square sense for AWGN channels.

Theorem 1. *A necessary condition for the stability of the system (1) over communication channel (2) in the sense of* $\lim_{k\to\infty} E[(\mathbf{x}_k)^{tr}(\mathbf{x}_k)|\zeta^{\lfloor mk \rfloor}] \overset{a.s}{<} \infty$ *is presented as* $mC \geq \sum_{\{j=1,|\lambda_j(A)|>1\}} \ln|\lambda_j(A)|$, *where* C *is the channel capacity.*

Proof: Since the stable eigenvalues of system have no effect on the stability [3],without loss of generality, we assume that all eigenvalues are outside the unit circle. At first, let we use X, F, V, Z, U and W as the random variables associated with, respectively, the system state, channel input, channel output, channel CSI, system input, and process noise. Subsequently, we follow the similar steps as taken in the proof of Lemma 1 in [24] to obtain the following inequity in the case of mismatch factor m:

$$N_{\zeta^i}(X_{\lfloor \frac{i}{m} \rfloor}|V^i) \geq \exp(-\frac{2}{d}C_i)N_{\zeta^{i-1}}(X_{\lfloor \frac{i}{m} \rfloor}|V^{i-1}), \tag{4}$$

where $C_i \triangleq \frac{1}{2}\ln(1+\frac{\zeta_i^2 P}{N_0/2})$ and $N_{\zeta^i}(X_{\lfloor \frac{i}{m} \rfloor}|V^i)$ denotes the conditional averaged entropy power of $X_{\lfloor \frac{i}{m} \rfloor}$. It should be noted that conditional averaged entropy power is defined as $N_{\zeta^i}(X_{\lfloor \frac{i}{m} \rfloor}|V^i) \triangleq \frac{1}{2\pi e}\exp(\frac{2}{d}E_{V^i|\zeta^i}[h(X_{\lfloor \frac{i}{m} \rfloor}|V^i = v^i, Z_i = \zeta^i])$, where $h(.|.)$ denote the conditional differential entropy and $E_{V^i|\zeta^i}$ denotes the expected value function only over random variable V^i conditioning on the event $Z_i = \zeta^i$. Using equation (1) results in:

$$\begin{aligned} h(X_{\lfloor \frac{i}{m} \rfloor}|V^{i-1} = v^{i-1}, Z^{i-1} = \zeta^{i-1}] =& h\Bigg(A^{\lfloor \frac{i}{m} \rfloor - \lfloor \frac{i-1}{m} \rfloor}X_{\lfloor \frac{i-1}{m} \rfloor} + B\sum_{t=\lfloor \frac{i-1}{m} \rfloor}^{\lfloor \frac{i}{m} \rfloor -1} A^{\lfloor \frac{i}{m} \rfloor -1-t}U_t \\ &+ \sum_{t=\lfloor \frac{i-1}{m} \rfloor}^{\lfloor \frac{i}{m} \rfloor -1} A^{\lfloor \frac{i}{m} \rfloor -1-t}W_t \Bigg| V^{i-1} = v^{i-1}, Z^{i-1} = \zeta^{i-1}\Bigg), \\ &\overset{(a)}{\geq} (\lfloor i/m \rfloor - \lfloor (i-1)/m \rfloor)\ln|det(A)| + h(X_{\lfloor \frac{i-1}{m} \rfloor}|V^{i-1} = v^{i-1}, Z^{i-1} = \zeta^{i-1}]), \end{aligned} \tag{5}$$

where (a) follows due to the fact that u_t is a function of (v^t, ζ^t), and furthermore, W_t , $t \in \{\lfloor \frac{i-1}{m} \rfloor, \ldots, \lfloor \frac{i}{m} \rfloor - 1\}$ and $X_{\lfloor \frac{i}{m} \rfloor}$ are two independent random variables. Therefore, using inequalities (4) and (5) results in:

$$N_{\zeta^i}(X_{\lfloor \frac{i}{m} \rfloor}|V^i) \geq \exp\left(-\frac{2}{d}\big(C_i - (\lfloor \frac{i}{m} \rfloor - \lfloor \frac{i-1}{m} \rfloor)\ln|det(A)|\big)\right) N_{\zeta^{i-1}}(X_{\lfloor \frac{i-1}{m} \rfloor}|V^{i-1}). \tag{6}$$

The recursive solution leads to inequality $N_{\zeta^i}(X_{\lfloor \frac{i}{m} \rfloor}|V^i) \geq \exp\Bigg(-\frac{2}{d}\big(\sum_{t=1}^{i} C_t - \lfloor \frac{i}{m} \rfloor \ln|det(A)|\big)\Bigg) \times N_{\zeta_0}(X_0|V^0)$. By preposition II.I in [31] and the assumption of theorem, its conclude that condition $\lim_{i\to\infty} N_{\zeta^i}(X_{\lfloor \frac{i}{m} \rfloor}|V^i) \leq \lim_{i\to\infty} E[(\mathbf{X}_{\lfloor \frac{i}{m} \rfloor})^{tr}(\mathbf{X}_{\lfloor \frac{i}{m} \rfloor})|Z^i = \zeta^i] \overset{a.s}{<} \infty$ holds for this system which results in $\lim_{i\to\infty} \frac{1}{i}\sum_{t=1}^{i} C_t - \frac{1}{i}\lfloor \frac{i}{m} \rfloor \ln|det(A)| \overset{a.s}{\geq} 0$. Using the law of large number in [32] leads to the following inequality as the necessary condition:

$$\lim_{i\to\infty} \frac{1}{i}\sum_{t=1}^{i} C_t - \frac{1}{i}\lfloor\frac{i}{m}\rfloor \ln|det(A)|) \overset{a.s}{=} E[C_i] - \frac{1}{m}\ln|det(A)| \geq 0. \tag{7}$$

Hence, the stability necessary condition in the sense of $\lim_{k\to\infty} E[(\mathbf{x}_k)^{tr}(\mathbf{x}_k)|\zeta^{\lfloor mk\rfloor}] \overset{a.s}{<} \infty$ is reduced to $mC = mE[C_i] \geq \sum_{\{j=1,|\lambda_j(A)|>1\}} \ln|\lambda_j(A)|$. ■

3.2. Sufficient conditions for stability

First, we propose two classes of encoder-decoder:

(1) The encoders that map $(\mathbf{x}_0, v^{i-1}, \zeta^{i-1}) \to f_i$ for every $i \in \{0, 1, \ldots\}$, and the decoders that map $(\mathbf{u}^{k-1}, v^{\lfloor mk\rfloor}, \zeta^{\lfloor mk\rfloor}) \to \hat{\mathbf{x}}_{k|\lfloor mk\rfloor}$ at every time instant kT_s, where $k \in \{0, 1, \ldots\}$. This can be used whenever the ambiguity in the states of system is only due to the ambiguity in the initial state. Hence, it is possible to guarantee reliable estimation by focusing on reconstruction of the initial state at the receiver and sending them to the transmitter throughout all communication steps. For scalar systems, we can describe this coding scheme, in general, as follows. At every time slot i, $f_i = T_1(i)(x_0 - \hat{x}_{0|i-1})$ is transmitted (for simplicity we assume $\hat{x}_{0|-1} = 0$). Upon receiving v_i, $\hat{x}_{0|i} = \hat{x}_{0|i-1} + R_1(i)v_i$ is computed and sent back to the transmitter, where $\big(T_1(i), R_1(i)\big)$ are the gains associated to the first class of encoder-decoder that would be defined in different encoding designs. Furthermore, the decoder obtains $\hat{x}_{k|\lfloor mk\rfloor}$ as the input of controller at every time instant kT_s.

(2) The encoders that map $(\mathbf{x}_{\lfloor \frac{i}{m}\rfloor}, v^{i-1}, \zeta^{i-1}) \to f_i$ and decoders which maps $(\mathbf{u}^{k-1}, v^{\lfloor mk\rfloor}, \zeta^{\lfloor mk\rfloor}) \to \hat{\mathbf{x}}_{k|\lfloor mk\rfloor}$. For scalar systems, we can describe this coding scheme as follows. In contrast with the first class, in this case, at every time slot i, $f_i = T_2(i)(x_{\lfloor \frac{i}{m}\rfloor} - \hat{x}_{\lfloor \frac{i}{m}\rfloor|i-1})$ is transmitted (for simplicity we assume $\hat{x}_{0|-1} = 0$). Upon receiving v_i, $\hat{x}_{\lfloor \frac{i+1}{m}\rfloor|i}$ is computed as follows and sent back to the transmitter.

$$\hat{x}_{\lfloor \frac{i}{m}\rfloor|i} = \hat{x}_{\lfloor \frac{i}{m}\rfloor|i-1} + R_2(i)v_i, \tag{8}$$

$$\hat{x}_{\lfloor \frac{i+1}{m}\rfloor|i} = A^{\lfloor \frac{i+1}{m}\rfloor - \lfloor \frac{i}{m}\rfloor}\hat{x}_{\lfloor \frac{i}{m}\rfloor|i} + \sum_{j=\lfloor \frac{i}{m}\rfloor}^{\lfloor \frac{i+1}{m}\rfloor - 1} A^{\lfloor \frac{i+1}{m}\rfloor - 1 - j}Bu_j, \tag{9}$$

where $\big(T_2(i), R_2(i)\big)$ are the gains associated to the second class of encoder-decoder that would be defined in different encoding designs. In what follows, we discuss different cases and obtain sufficient conditions for stability of each case.

(A) Scalar discrete-time system without process noise over AWGN channel:

For achieving the sufficient condition in this case, we have incorporated Schalkwijk-Kailath (S-K) scheme based on Maximum Likelihood (ML) estimation. It should be noted that for the similar system model with $m = 1$, [12] have also incorporated S-K coding scheme based on Minimum Mean Square Error (MMSE) estimation in their achievability proof.

Remark 2. *Using ML estimator in this problem provides the opportunity of designing an encoder/decoder which is not sensitive to the Probability Density Function (PDF) of the initial state. Generally, all the proposed coding structures in this paper are applicable to a bounded variance initial state with arbitrary PDF.*

Theorem 2. *Control/communication system of Fig. 1 consisting of scalar discrete-time plant (1) without process noise and discrete-time AWGN channel (3) with capacity C is asymptotically stabilizable in the mean square sense if* $\ln(|A|) < mC = \frac{m}{2}\ln(1 + \frac{P}{N_0/2})$.

Proof: It is possible to prove this theorem using both classes of coding scheme. However, for simplicity, we employ the first class to obtain a sufficient condition. Based on the encoder-decoder design in [33] and defining the parameter α as

$$\alpha = \exp(C) = (1 + \frac{P}{N_0/2})^{\frac{1}{2}}, \tag{10}$$

the encoder-decoder gains are chosen as follows:

$$(T_1(i), R_1(i)) = \begin{cases} (\alpha, \alpha^{-1}) & \text{if } i = 0, \\ \left(\alpha^i(\alpha^2-1)^{\frac{1}{2}}, \alpha^{-i}\frac{(\alpha^2-1)^{\frac{1}{2}}}{\alpha^2}\right) & \text{if } i \geq 1. \end{cases}$$

The recursive solution is given by [33]:

$$\hat{x}_{0|i} \sim N(x_0, \frac{N_0/2}{(\alpha^2)^{i+1}}), \tag{11}$$

where $\hat{x}_{0|i}$ is ML estimate of x_0 given $\{v_0, v_1, \ldots, v_i\}$ and $x_0 - \hat{x}_{0|i}$ is only a function of channel noise n_j, $j \in \{0, 1, \ldots, i\}$ [33]. The average transmit power of this scheme is

$$P_{av} = \lim_{N\to\infty} P_{av}(N) = \lim_{N\to\infty} \frac{1}{N} E[\alpha^2(x - \hat{x}_{0|-1})^2 + \sum_{i=1}^{N-1} (\alpha^i(\alpha^2-1)^{\frac{1}{2}})^2 (x - \hat{x}_{0|i-1})^2]. \tag{12}$$

Assuming bounded variance for x_0 and $E[x_0^2] = \Psi$, we have the following relations:

$$P_{av}(N) = \frac{\alpha^2\Psi}{N} + \frac{N-1}{N}\frac{N_0}{2}(\alpha^2 - 1), \tag{13}$$

hence, asymptotically $P_{av} = P$ and the channel power constraint is satisfied. The decoder should estimate x_k given $v_0, \ldots, v_{\lfloor mk \rfloor}$ at time instant kT_s. Solving differential equation (1) (under the assumption of $G = 0$) leads to:

$$x_k = A^k x_0 + \sum_{j=1}^{k-1} A^{k-1-j} B u_j, \tag{14}$$

where the sequence $\{u_j,\ 0 \leq j \leq k-1\}$ is a function of $v_0, \ldots, v_{\lfloor mk \rfloor}$ and is available at the receiver. Hence, we finally arrive at the desired estimate as follows:

$$\hat{x}_{k|\lfloor mk \rfloor} = A^k \hat{x}_{0|\lfloor mk \rfloor} + \sum_{j=0}^{k-1} A^{k-1-j} B u_j. \tag{15}$$

Using equations (14) and (15), we define $e_{k|\lfloor mk \rfloor}$ as $e_{k|\lfloor mk \rfloor} \triangleq x_k - \hat{x}_{k|v_0,\ldots,v_{\lfloor mk \rfloor}}$. Taking the expectation yields:

$$E[e^2_{k|\lfloor mk \rfloor}] = A^{2k} E[(x_0 - \hat{x}_{0|\lfloor mk \rfloor})^2]. \tag{16}$$

If we choose control gain for system (1) as $\mathcal{G}_k = -\frac{A}{B}$, then using equation (11) results in:

$$E[x^2_{k+1}] = A^2 E[e^2_{k|\lfloor mk \rfloor}] = A^{2(k+1)} \frac{N_0/2}{\alpha^{2(\lfloor mk \rfloor + 1)}}. \tag{17}$$

Hence, system (1) is stable if $\lim_{k\to\infty} \frac{|A|^k}{\alpha^{\lfloor mk \rfloor}} \to 0$ or equivalently if $\ln(|A|) < m\ln(\alpha)$.
By substituting equation $\alpha = \exp(C)$ in the above condition, we obtain the sufficient condition for stability as $\ln(|A|) < mc = \frac{m}{2}\ln(1+\frac{P}{N_0/2})$. Consequently, the tightness of sufficient condition for this case (i.e., A) is proven. ■

(B) Scalar discrete-time system without process noise over AWGN fading channel:

We address the control/communication system consisting of AWGN fading channel with noiseless feedback which is more common for wireless channels. We present the results in the following theorem.

Theorem 3. *Control/communication system of Fig. 1 consisting of scalar discrete-time plant (1) without process noise and discrete-time AWGN fading channel (2) with channel capacity C is asymptotically stabilizable (in the sense that $E[x_{k+1}^2|\zeta^{\lfloor mk \rfloor}]$ converges to zero almost surely) if $\ln(|A|) < mC = \frac{m}{2}E_{\zeta_i}(\ln(1+\frac{\zeta_i^2 P}{N_0/2}))$.*

Proof: As the system has no process noise, we can design an encoder of the first class. Based on the encoder/decoder design in [34], we propose the following parameters for designing the encoder and decoder:

$$(T_1(i), R_1(i)) = \begin{cases} (\rho, -\frac{b(\zeta_0)}{\rho a(\zeta_0)}) & \text{if } i = 0, \\ \left(\rho\prod_{k=0}^{i-1} a(\zeta_k), -\frac{b(\zeta_i)}{\rho\prod_{k=0}^{i} a(\zeta_k)}\right) & \text{if } i \geq 1, \end{cases}$$

where ρ which is a constant number, should be selected such that $E[f_0^2] < P$. It is proven that if $E[f_0^2] < P$, the average input power is not larger than P [34]. Note that the channel state ζ_i is available and used at the receiver side at time i to compute $\hat{x}_{0|i}$, while it is used at the transmitter side with one delay to send f_{i+1}. Hence, the recursion of f is:

$$\begin{aligned} f_{i+1} &= a(\zeta_i)f_i + b(\zeta_i)v_i, \\ v_i &= \zeta_i f_i + n_i. \end{aligned} \tag{18}$$

The parameters $a(\zeta_i)$ and $b(\zeta_i)$ are chosen as follows:

$$a(\zeta_i) = \sqrt{(\frac{P}{N_0/2})\zeta_i^2 + 1}, \quad b(\zeta_i) = -\frac{\frac{P}{N_0/2}\zeta_i}{\sqrt{(\frac{P}{N_0/2})\zeta_i^2 + 1}}.$$

Equation (18) and the same steps in [34] concludes that channel input satisfies power constraint $P_{av} \leq P$ and:

$$(x_0 - \hat{x}_{0|\lfloor mk \rfloor}|\zeta^{\lfloor mk \rfloor}, x_0) \sim N((\prod_{k=0}^{\lfloor mk \rfloor} a(\zeta_k))^{-2}x_0, (\prod_{k=0}^{\lfloor mk \rfloor} a(\zeta_k))^{-2}\frac{\sigma_f^2}{\rho^2})$$

where $\sigma_f^2 < (\lfloor mk \rfloor + 1)P$. As explained earlier, by choosing control gain $\mathcal{G}_k = -\frac{A}{B}$, we have,

$$E[x_{k+1}^2|\zeta^{\lfloor mk \rfloor}] \leq \frac{A^{2(k+1)}}{\rho^2(\prod_{l=0}^{\lfloor mk \rfloor} a(\zeta_l))^2}\Big(\frac{\rho^2\psi_0}{(\prod_{l=0}^{\lfloor mk \rfloor} a(\zeta_l))^2} + (\lfloor mk \rfloor + 1)P\Big).$$

Therefore, the sufficient condition for the stability can be presented as $\lim_{k\to\infty} \frac{A^{2(k+1)}}{\rho^2(\prod_{l=0}^{\lfloor mk \rfloor} a(\zeta_l))^2}(\lfloor mk \rfloor + 1)P \overset{a.s}{=} 0$. Note that $\lim_{k\to\infty} \frac{1}{\lfloor mk \rfloor + 1}\sum_{l=0}^{\lfloor mk \rfloor} \ln(a(\zeta_l)) \overset{a.s}{=} \ln(\bar{a})$ (using the law of large number), where

$\bar{a}$ is defined as $\bar{a} \triangleq \prod_{l=1}^{s} a(\zeta_l)^{\gamma_l}$. Hence, if $\ln(|A|) < m\ln(\bar{a}) = \frac{m}{2}E_{\zeta_i}(\ln(1+\frac{\zeta_i^2 P}{N_0/2})) = mC$, the aforementioned condition is satisfied and the system is stable. ■

(C) Scalar discrete-time system with process noise over AWGN channel:

For this case, the system ambiguity is due to both initial state and process noise; hence, the system cannot be stable if we only transmit information about initial state over the channel. Therefore, the coding scheme of second class is applied to this case.

Theorem 4. *Control/communication system of Fig. 1 consisting of scalar discrete-time plant (1) and discrete-time AWGN channel (3) with capacity C is bounded asymptotically stabilizable in the mean square sense if* $\ln(|A|) < mC = \frac{m}{2}\ln(1+\frac{P}{N_0/2})$.

Proof: We design an encoder of the second class through adjusting parameters as follows:

$$(T_2(i), R_2(i)) = \begin{cases} (\alpha, \alpha^{-1}) & \text{if } i = 0 \\ \left(d(i)(\alpha^2-1)^{\frac{1}{2}}, d(i)^{-1}\frac{(\alpha^2-1)^{\frac{1}{2}}}{\alpha^2}\right) & \text{if } i \geq 1 \end{cases} \tag{19}$$

where α is the same as in (10) and $d(i)$ is chosen such that the average power remains the same as in (13), i.e.,

$$d(i) = \frac{\sqrt{N_0/2}}{\sqrt{E[(x_{\lfloor \frac{i}{m} \rfloor} - \hat{x}_{\lfloor \frac{i}{m} \rfloor | i-1})^2]}}. \tag{20}$$

From differential equation (1), we have

$$x_{\lfloor \frac{i+1}{m} \rfloor} = A^{\lfloor \frac{i+1}{m} \rfloor - \lfloor \frac{i}{m} \rfloor} x_{\lfloor \frac{i}{m} \rfloor} + \sum_{j=\lfloor \frac{i}{m} \rfloor}^{\lfloor \frac{i+1}{m} \rfloor - 1} A^{\lfloor \frac{i+1}{m} \rfloor - 1 - j} B u_j + \sum_{j=\lfloor \frac{i}{m} \rfloor}^{\lfloor \frac{i+1}{m} \rfloor - 1} A^{\lfloor \frac{i+1}{m} \rfloor - 1 - j} G w_j. \tag{21}$$

Combining (8), (9), and (19) leads to the following equation for $i \geq 1$:

$$\begin{aligned} \hat{x}_{\lfloor \frac{i+1}{m} \rfloor | i} =& A^{\lfloor \frac{i+1}{m} \rfloor - \lfloor \frac{i}{m} \rfloor}(\hat{x}_{\lfloor \frac{i}{m} \rfloor | i-1} + d(i)^{-1}\frac{(\alpha^2-1)^{\frac{1}{2}}}{\alpha^2}\{(\alpha^2-1)^{\frac{1}{2}} d(i)(x_{\lfloor \frac{i}{m} \rfloor} - \hat{x}_{\lfloor \frac{i}{m} \rfloor | i-1}) + n_i\}) \\ &+ \sum_{j=\lfloor \frac{i}{m} \rfloor}^{\lfloor \frac{i+1}{m} \rfloor - 1} A^{\lfloor \frac{i+1}{m} \rfloor - 1 - j} B u_j. \end{aligned} \tag{22}$$

Subtracting (22) from (21) results in the following differential equation:

$$\begin{aligned} e_{\lfloor \frac{i+1}{m} \rfloor | i} &= \frac{A^{\lfloor \frac{i+1}{m} \rfloor - \lfloor \frac{i}{m} \rfloor}}{\alpha^2} e_{\lfloor \frac{i}{m} \rfloor | i-1} - d(i)^{-1} A^{\lfloor \frac{i+1}{m} \rfloor - \lfloor \frac{i}{m} \rfloor}\frac{(\alpha^2-1)^{\frac{1}{2}}}{\alpha^2} n_i + \sum_{j=\lfloor \frac{i}{m} \rfloor}^{\lfloor \frac{i+1}{m} \rfloor - 1} A^{\lfloor \frac{i+1}{m} \rfloor - 1 - j} G w_j. \\ e_{\lfloor \frac{1}{m} \rfloor | 0} &= A^{\lfloor \frac{1}{m} \rfloor}\frac{n_0}{\alpha} + \sum_{j=0}^{\lfloor \frac{1}{m} \rfloor - 1} A^{\lfloor \frac{1}{m} \rfloor - 1 - j} G w_j \end{aligned} \tag{23}$$

It is clear from the above equation that $e_{\lfloor \frac{i}{m} \rfloor | i-1}$ is independent of n_i and w_j, for $j \in \{\lfloor \frac{i}{m} \rfloor, \ldots, \lfloor \frac{i+1}{m} \rfloor - 1\}$. Furthermore, given the assumption that channel and process noise are independent, $E[e^2_{\lfloor \frac{i+1}{m} \rfloor | i}]$

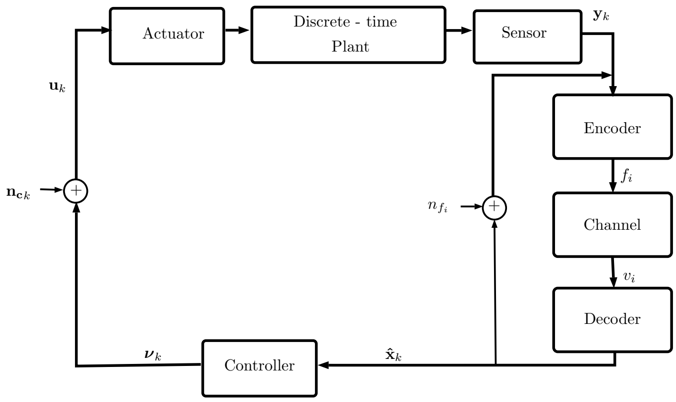

Figure 2: The more realistic control/communication system over AWGN channel

is calculated as follows:

$$E[e^2_{\lfloor \frac{i+1}{m} \rfloor|i}] = (\frac{A^{\lfloor \frac{i+1}{m} \rfloor - \lfloor \frac{i}{m} \rfloor}}{\alpha^2})^2 E[e^2_{\lfloor \frac{i}{m} \rfloor|i-1}] + d(i)^{-2}(A^2)^{\lfloor \frac{i+1}{m} \rfloor - \lfloor \frac{i}{m} \rfloor} \frac{(\alpha^2-1)}{\alpha^4} \frac{N_0}{2} + \sum_{j=\lfloor \frac{i}{m} \rfloor}^{\lfloor \frac{i+1}{m} \rfloor - 1} (A^2)^{\lfloor \frac{i+1}{m} \rfloor - 1 - j} G^2,$$

$$E[e^2_{\lfloor \frac{1}{m} \rfloor|0}] = A^{2\lfloor \frac{1}{m} \rfloor} \frac{N_0}{2\alpha^2} + \sum_{j=0}^{\lfloor \frac{1}{m} \rfloor - 1} (A^2)^{\lfloor \frac{1}{m} \rfloor - 1 - j} G^2 \tag{24}$$

By substituting (20) in (24), we finally arrive at the following equations for $E[e^2_{\lfloor \frac{i+1}{m} \rfloor|i}]$ and $i \geq 1$:

$$E[e^2_{\lfloor \frac{i+1}{m} \rfloor|i}] = (\frac{A^{\lfloor \frac{i+1}{m} \rfloor - \lfloor \frac{i}{m} \rfloor}}{\alpha})^2 E[e^2_{\lfloor \frac{i}{m} \rfloor|i-1}] + \sum_{j=\lfloor \frac{i}{m} \rfloor}^{\lfloor \frac{i+1}{m} \rfloor - 1} (A^2)^{\lfloor \frac{i+1}{m} \rfloor - 1 - j} G^2.$$

Clearly, if $\ln(|A|) < m\ln(\alpha) = \frac{m}{2}\ln(1+\frac{P}{N_0/2}) = mC$, then $E[e^2_{\lfloor \frac{i}{m} \rfloor|i}]$ is asymptotically bounded. Under this conditon and by choosing controller gain as $\mathcal{G}_k = -\frac{A}{B}$, $E[x^2_{k+1}]$ is asymptotically bounded, since

$$\lim_{k\to\infty} E[x^2_{k+1}] = \lim_{k\to\infty} (A^2)^{k+1-\lfloor \frac{\lfloor mk \rfloor}{m} \rfloor} E[e^2_{\lfloor \frac{\lfloor mk \rfloor}{m} \rfloor|\lfloor mk \rfloor}] + \sum_{j=\lfloor \frac{\lfloor mk \rfloor}{m} \rfloor}^{k} (A^2)^{k-j} G^2 < \infty. \tag{25}$$

Hence, the system is stable in the mean square sense. This completes the proof. ■

(D) Scalar discrete-time system over AWGN channel with noisy links:

As mentioned earlier, this important case is one of key contributions of this paper which is achieved through proper adoption of the second coding scheme. It is important to note that in reality, some level of noise exists in the feedback link between transmitter and receiver and also in the link between the controller and actuator as shown in Fig. 2. The following theorem is about the sufficient condition for the stability in such scenarios.

Theorem 5. *Control/communication system of Fig. 2 consisting of scalar discrete-time plant (1) and discrete-time AWGN channel (3) with capacity C is bounded asymptotically stabilizable in the mean square sense if* $\ln(|A|) < mC = \frac{m}{2}\ln(1+\frac{P}{N_0/2})$.

Proof: Assume that n_{f_i} and n_{c_k} in Fig. 2 are white Gaussian noise with variances N_f and N_c, respectively. We assume that the receiver side has access to the output sequence of controller (i.e.,

ν^{k-1}) where $u_k = \nu_k + n_{c_k}$. Furthermore, transmitter has noisy access to $\hat{x}_{\lfloor \frac{i+1}{m} \rfloor | i}$. Using the encoder and decoder from the second class with parameters (19) and $d(i) = \frac{\sqrt{N_0/2}}{\sqrt{E[(x_{\lfloor \frac{i}{m} \rfloor} - \hat{x}_{\lfloor \frac{i}{m} \rfloor | i-1})^2] + N_f}}$, keeps the average power the same as (13). Subsequently, we can show that $E[e^2_{k|\lfloor mk \rfloor}]$ still converges but now to a higher bound. Considering noisy links, we have the following relations:

$$\hat{x}_{\lfloor \frac{i+1}{m} \rfloor | i} = A^{\lfloor \frac{i+1}{m} \rfloor - \lfloor \frac{i}{m} \rfloor} \hat{x}_{\lfloor \frac{i}{m} \rfloor | i} + \sum_{j=\lfloor \frac{i}{m} \rfloor}^{\lfloor \frac{i+1}{m} \rfloor - 1} A^{\lfloor \frac{i+1}{m} \rfloor - 1 - j} B \nu_j. \tag{26}$$

Using (8), (19), and subtracting ($\hat{x}_{\lfloor \frac{i+1}{m} \rfloor | i}$) from $x_{\lfloor \frac{i+1}{m} \rfloor}$ leads to the following equation for $i \geq 1$:

$$\begin{aligned} x_{\lfloor \frac{i+1}{m} \rfloor} - \hat{x}_{\lfloor \frac{i+1}{m} \rfloor | i} = & A^{\lfloor \frac{i+1}{m} \rfloor - \lfloor \frac{i}{m} \rfloor} (x_{\lfloor \frac{i}{m} \rfloor} - \hat{x}_{\lfloor \frac{i}{m} \rfloor | i-1} - d(i)^{-1} \frac{(\alpha^2 - 1)^{\frac{1}{2}}}{\alpha^2} v_i) \\ & + \sum_{j=\lfloor \frac{i}{m} \rfloor}^{\lfloor \frac{i+1}{m} \rfloor - 1} A^{\lfloor \frac{i+1}{m} \rfloor - 1 - j} B (u_j - \nu_j) + \sum_{j=\lfloor \frac{i}{m} \rfloor}^{\lfloor \frac{i+1}{m} \rfloor - 1} A^{\lfloor \frac{i+1}{m} \rfloor - 1 - j} G w_j. \end{aligned} \tag{27}$$

Note that although the receiver sends back $\hat{x}_{\lfloor \frac{i}{m} \rfloor | i-1}$ to the transmitter, but the transmitter receives it as $\hat{x}_{\lfloor \frac{i}{m} \rfloor | i-1} + n_{f_{i-1}}$. Therefore, $f_i = T_2(i)(x_{\lfloor \frac{i}{m} \rfloor} - \hat{x}_{\lfloor \frac{i}{m} \rfloor | i-1} - n_{f_{i-1}})$, and (27) can be rewritten as follows for $i \geq 1$:

$$\begin{aligned} e_{\lfloor \frac{i+1}{m} \rfloor | i} = & \frac{A^{\lfloor \frac{i+1}{m} \rfloor - \lfloor \frac{i}{m} \rfloor}}{\alpha^2} e_{\lfloor \frac{i}{m} \rfloor | i-1} - A^{\lfloor \frac{i+1}{m} \rfloor - \lfloor \frac{i}{m} \rfloor} (\frac{\alpha^2 - 1}{\alpha^2} n_{f_{i-1}} + d(i)^{-1} \frac{(\alpha^2 - 1)^{\frac{1}{2}}}{\alpha^2} n_i) \\ & + \sum_{j=\lfloor \frac{i}{m} \rfloor}^{\lfloor \frac{i+1}{m} \rfloor - 1} A^{\lfloor \frac{i+1}{m} \rfloor - 1 - j} B n_{c_j} + \sum_{j=\lfloor \frac{i}{m} \rfloor}^{\lfloor \frac{i+1}{m} \rfloor - 1} A^{\lfloor \frac{i+1}{m} \rfloor - 1 - j} G w_j, \end{aligned} \tag{28}$$

where $e_{\lfloor \frac{1}{m} \rfloor | 0} = A^{\lfloor \frac{1}{m} \rfloor} \frac{n_0}{\alpha} + \sum_{j=0}^{\lfloor \frac{1}{m} \rfloor - 1} A^{(\lfloor \frac{1}{m} \rfloor - 1 - j)} (B n_{c_j} + G w_j)$. Note that $E[e_{\lfloor \frac{i}{m} \rfloor | i-1} n_{f_{i-1}}] = 0$. Therefore, the error variance is satisfied in the following differential equation for $i \geq 1$:

$$\begin{aligned} E[e^2_{\lfloor \frac{i+1}{m} \rfloor | i}] = & \frac{A^{2(\lfloor \frac{i+1}{m} \rfloor - \lfloor \frac{i}{m} \rfloor)}}{\alpha^2} E[e^2_{\lfloor \frac{i}{m} \rfloor | i-1}] + (A^{2(\lfloor \frac{i+1}{m} \rfloor - \lfloor \frac{i}{m} \rfloor)} - \frac{A^{2(\lfloor \frac{i+1}{m} \rfloor - \lfloor \frac{i}{m} \rfloor)}}{\alpha^2}) N_f \\ & + \sum_{j=\lfloor \frac{i}{m} \rfloor}^{\lfloor \frac{i+1}{m} \rfloor - 1} A^{2(\lfloor \frac{i+1}{m} \rfloor - 1 - j)} B^2 N_c + \sum_{j=\lfloor \frac{i}{m} \rfloor}^{\lfloor \frac{i+1}{m} \rfloor - 1} A^{2(\lfloor \frac{i+1}{m} \rfloor - 1 - j)} G^2, \end{aligned} \tag{29}$$

where $E[e^2_{\lfloor \frac{1}{m} \rfloor | 0}] = A^{2\lfloor \frac{1}{m} \rfloor} \frac{N_0}{2\alpha^2} + \sum_{j=0}^{\lfloor \frac{1}{m} \rfloor - 1} A^{2(\lfloor \frac{1}{m} \rfloor - 1 - j)} (B^2 N_c + G^2)$. Consequently, if the communication and control signals are sent back through noisy feedback, as long as $\ln(|A|) < m \ln(\alpha) = mC$, the value of $E[e^2_{\lfloor \frac{i}{m} \rfloor | i}]$ is asymptotically bounded and by choosing proper controller gain (similar to what is explained in scenario C), the system is stable in the bounded asymptotic sense . ■

As noted earlier, transmission of signals through finite number of quantization bits is a practical approach. It is easy to observe that modeling white quantization error as additive Gaussian noise, leads to the same results.

Remark 3. *In the presence of noisy channel feedback, the stability is still achievable under condition* $\ln(|A|) < m \ln(\alpha) = mC$ *with the difference that the variance of state signal converges to a* *higher level rather than the case of perfect feedback. When the noise variances of communication feedback and controller-actuator links tend to zero, the results are the same as the previous values obtained in the case (C). In the same scenarios, asymptotic stability can be achieved in the systems without process noise.*

4. Extension to the vector case

So far, we have treated the scalar systems. Extensions of the results to the vector cases are obtained through the Time Division Multiplexing (TDM), as described in the following. The results for discrete-time multi-dimension plant over AWGN channel are given in the following theorem.

Theorem 6. *Control/communication system of Fig. 2 consisting of discrete-time plant (1) and discrete-time AWGN channel (3) is bounded asymptotically stabilizable (or asymptotically stabilizable in the special case of $N_f = 0$, $N_C = 0$, and $G = 0$) in the mean square sense if $\sum_{\{j;|\lambda_j(A)|>1\}} \ln|\lambda_j(A)| < mC = \frac{m}{2}\ln(1 + \frac{P}{N_0/2})$, where m is the mismatch factor.*

Proof: We can extend our results to multi-dimensional systems using TDM approach. This approach is also used in [35] to extend their one-dimensional results to n-dimensional systems. We explain such extension in more detail.
The stable part of A has no effect on the stability condition, consequently, without loss of generality, we assume that all eigenvalues of A are outside the unit circle [3].
For $A \in \mathbb{R}^{d\times d}$ there exists a real-valued non-singular matrix Φ and a real-valued matrix Γ such that $\Phi A \Phi^{-1} = \Gamma = diag[J_1, \ldots, J_n]$; where $J_l, l = 1, \ldots, n$, is a Jordan block with geometric multiplicity d_l while eigenvalues of Γ and A are the same.
We define $H = diag[H_1, \ldots, H_n]$ where H_l is associated with one of the Jordan blocks J_l. For Jordan block associated with real eigenvalues, $H_l = I_{d_l}$ and for $\lambda_j = \bar{\rho}(\cos(\theta) \pm j\sin(\theta))$, we have $H_l = [r(\theta)^{-1}, \ldots, r(\theta)^{-1}]$ where $r(\theta) \triangleq \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \cos(\theta) & -\sin(\theta) \end{bmatrix}$. It is shown in [3] that Γ and any power of the matrix H commute (i.e., $H^k \Gamma H^{-k} = \Gamma$). Consequently, a new variable $\mathbf{z}_k = H^k \Phi \mathbf{x}_k$ is defined to rewrite the equation (1) as:

$$\begin{aligned} \mathbf{z}_{k+1} &= H\Gamma \mathbf{z}_k + \widetilde{\mathbf{u}}_k + \widetilde{\mathbf{w}}_k \\ \mathbf{u}_k &= B^+(H^{k+1}\Phi)^{-1}\widetilde{\mathbf{u}}_k; \\ \widetilde{\mathbf{w}}_k &= H^{k+1}\Phi G \mathbf{w}_k; \\ \widetilde{\mathbf{w}}_k &\sim N(\underline{0}_d, \widetilde{Q}_k), \end{aligned} \tag{30}$$

where $\widetilde{\mathbf{w}}_k$ is the white noise. In addition, since the magnitude of eigenvalues of H is equal to one, $\widetilde{Q}_k = H^{k+1}\Phi G (H^{k+1}\Phi G)^{tr}$ is bounded (i.e. $\widetilde{Q}_k \leq \bar{Q}$).
Note that $H\Gamma$ is an upper triangular matrix with real-valued eigenvalues, whose magnitudes are the same as the magnitude of the corresponding eigenvalues of Γ defined by λ_j, $j \in \{1, 2, \ldots, d\}$. Without loss of generality, we assume these eigenvalues are positive and based on Jordan properties, they are ordered as $|\lambda_1| \leq \ldots \leq |\lambda_d|$. Given that $H^t\Phi$ is an invertible matrix, stability of $\mathbf{z}_k$ and $\mathbf{x}_k$ are equivalent. Hence, it is adequate to prove that condition $\sum_{\{j;|\lambda_j(A)|>1\}} \ln|\lambda_j(A)| < \frac{m}{2}\ln(1 + \frac{P}{N_0/2})$ is a sufficient condition for stability of system (30).We prove this theorem in two parts.

Part 1: As mentioned earlier, in the case of ($N_f = 0$, $N_c = 0$, and $G = 0$), the whole ambiguity in the system only depends on the initial state. In this case, one can design an encoder-decoder of the first class. Note that since $\mathbf{z}_0 \in \mathbb{R}^d$, we can consider $\mathbf{z}_0$ as d scalar random variables. Therefore, it is sufficient that for each element of $\mathbf{z}_0$, the encoder-decoder used for the scalar case (A) is adopted. In order to apply such encoding/decoding scheme, we change the channel

into d parallel channels using a basic TDM approach. In fact, we sequentially assign q_j TDM time intervals to the jth dimension of $\mathbf{z}_0$ where $j \in \{1, 2, \ldots, d\}$. Therefore, a total number of $q = q_1 + ... + q_d$ TDM intervals is used for transmission of $\mathbf{z}_0$ over a period of qT_c.
It should be noted that $\mathbf{z}_0^{(j)} - \hat{\mathbf{z}}_{0|q-1}^{(j)}, j \in \{1, \ldots, d\}$ is only a function of channel noise $n_i, i \in \{\sum_{t=1}^{j-1} q_t, \ldots, \sum_{t=1}^{j} q_t - 1\}$ since, up to time $q-1$, $\mathbf{z}_0^{(j)}$ is only transmitted during these q_j intervals. Therefore, $\mathbf{z}_0^{(j)} - \hat{\mathbf{z}}_{0|q-1}^{(j)}$ and $\mathbf{z}_0^{(\ell)} - \hat{\mathbf{z}}_{0|q-1}^{(\ell)}$ for $j \neq \ell$ are the function of white Gaussian noise in different time intervals which results in their independency. Therefore, by definition $\mathbf{e}_{0|q-1}^{z} \triangleq \mathbf{z}_0 - \hat{\mathbf{z}}_{0|q-1}$, and we can write $\mathbf{e}_{0|q-1}^{z} \sim N(\underline{0}_d, \frac{N_0}{2} diag(\frac{1}{\alpha^{2q_1}}, \ldots, \frac{1}{\alpha^{2q_d}}))$.
The error covariance matrix of $\mathbf{z}_k$ is subsequently calculated as follows:

$$\begin{aligned} \mathbf{e}_{k|\lfloor mk \rfloor}^{z} &\triangleq \mathbf{z}_k - \hat{\mathbf{z}}_{k|\lfloor mk \rfloor} = (H\Gamma)^k (\mathbf{z}_0 - \hat{\mathbf{z}}_{0|\lfloor mk \rfloor}) \\ E[\mathbf{e}_{k|\lfloor mk \rfloor}^{z} (\mathbf{e}_{k|\lfloor mk \rfloor}^{z})^{tr}] &= (H\Gamma)^k D((H\Gamma)^k)^{tr}, \end{aligned} \tag{31}$$

where D is defined as a diagonal matrix whose the jth element is equal to $D_{jj} = \frac{N_0}{2} \frac{1}{(\alpha^2)^{\lfloor (\lfloor mk \rfloor + 1)/q \rfloor q_j + b_j(\lfloor mk \rfloor)}}$, where $b_j(i) \triangleq \min\{(mod(i+1, q) - \sum_{t=1}^{j-1} b_t(i)), q_j\}$.
Note that choosing controller gain as $\mathcal{G}_k = -(H\Gamma)$ for the system (30) results in $\widetilde{\mathbf{u}}_k = -H\Gamma \mathbf{z}_{k|\lfloor mk \rfloor}$ and therfeor, $\mathbf{z}_{k+1} = H\Gamma \mathbf{z}_k + \widetilde{\mathbf{u}}_k = H\Gamma \mathbf{e}_{k|\lfloor mk \rfloor}^{z}$, and therefore, by Definition 1, the system (30) is asymptotically stable in mean square sense if $trace\{E[\mathbf{e}_{k|\lfloor mk \rfloor}^{z} (\mathbf{e}_{k|\lfloor mk \rfloor}^{z})^{tr}]\}$ converges to zero.
In addition, note that from the above equation we have,

$$trace(E[\mathbf{e}_{k|\lfloor mk \rfloor}^{z} (\mathbf{e}_{k|\lfloor mk \rfloor}^{z})^{tr}]) = trace((H\Gamma)^k D^{\frac{1}{2}} ((H\Gamma)^k D^{\frac{1}{2}})^{tr}). \tag{32}$$

Therefore, for stability purposes, it is enough to find a condition under which the right hand side of the above equation tends to zero. In the above equation, $(H\Gamma)^k D^{\frac{1}{2}}$ is an upper triangular matrix with eigenvalues

$$\lambda\{(H\Gamma)^k\}_j \frac{N_0/2}{\alpha^{\lfloor (\lfloor mk \rfloor + 1)/q \rfloor q_j + b_j(\lfloor mk \rfloor)}} = |\lambda_j|^k \frac{N_0/2}{\alpha^{\lfloor (\lfloor mk \rfloor + 1)/q \rfloor q_j + b_j(\lfloor mk \rfloor)}}$$

along its diagonal. It can be shown that if all eigenvalues go to zero, then $(H\Gamma)^k D^{\frac{1}{2}}$ goes to the zero matrix (See Appendix A). Hence, If $\frac{|\lambda_j|}{\alpha^{(q_j/q)m}} < 1$ or equivalently $(\ln |\lambda_j| < m(q_j/q) \ln \alpha)$ for every $j \in \{1, 2, \ldots, d\}$, all eigenvalues go to zero; and hence, $\lim_{k \to \infty} trace(E[\mathbf{e}_{k|\lfloor mk \rfloor}^{z} (\mathbf{e}_{k|\lfloor mk \rfloor}^{z})^{tr}]) = 0$.
Therefore, in the special case of $N_f = 0$, $N_c = 0$, and $G = 0$, the sufficient condition for asymptotic stability is $(\ln |\lambda_j| < m(q_j/q) \ln \alpha)$ for every $j \in \{1, 2, \ldots, d\}$.

Part 2: In this part, we consider the effect of process noise, feedback channel noise and the noise in the controller-actuator link. Similar to the scalar case, we should design an encoder-decoder of the second class for bounded asymptotic stability in the mean square sense. In the first step, we view $\mathbf{z}_0$ as d scalar variables and apply the coding scheme in case (D) for each element of the vector $\mathbf{z}_0$ for finite time slots. In fact, we sequentially transmit $\mathbf{z}_0^{(j)}$ for q_j intervals for every j, $j \in \{1, 2, \ldots, d\}$. Therefore, by writing the equations (28) and (29) for $A = 1$, $B = 0$ and $G = 0$, it is proven that after $q = q_1 + \ldots + q_d$ time intervals, each element of the error vector $\mathbf{e}_{0|q-1}^{z}$ is only a function of channel noise and the noise of channel feedback in its corresponding time intervals. Therefore, Due to the whiteness assumption of the noise, the error covariance matrix $E[(\mathbf{e}_{0|q-1}^{z})(\mathbf{e}_{0|q-1}^{z})^{tr}] = \mathcal{P}(N_0/2)$ is diagonal where $\mathcal{P}(\chi)$ is defined as a diagonal matrix whose jth element and $j \in \{1, 2, \ldots, d\}$ is determined based on equation (29) as follows:

$$\{\mathcal{P}(\chi)\}_{jj} = \frac{1}{\alpha^{2q_j}} \chi + \sum_{\ell=0}^{q_j - 2} (\frac{1}{\alpha^2})^{q_j - 2 - \ell} (1 - \frac{1}{\alpha^2}) N_f, \tag{33}$$

where α is the same as in (10). Subsequently, the receiver calculates the following equation and sends it back to the transmitter.

$$\hat{\mathbf{z}}_{\lfloor \frac{q}{m} \rfloor|q-1} = H\Gamma^{\lfloor \frac{q}{m} \rfloor}\hat{\mathbf{z}}_{0|q-1} + \sum_{l=0}^{\lfloor \frac{q}{m} \rfloor -1} (H\Gamma)^{\lfloor \frac{q}{m} \rfloor - l - 1}(\widetilde{\mathbf{u}}_l - \mathbf{n_c}_l).$$

The error covariance matrix is obtained at the transmitter side as follows:

$$E[(\mathbf{e}^z_{\lfloor \frac{q}{m} \rfloor|q-1})(\mathbf{e}^z_{\lfloor \frac{q}{m} \rfloor|q-1})^{tr}] = (H\Gamma^{\lfloor \frac{q}{m} \rfloor})\mathcal{P}(\frac{N_0}{2})(H\Gamma^{\lfloor \frac{q}{m} \rfloor})^{tr} + \sum_{l=0}^{\lfloor \frac{q}{m} \rfloor -1} (H\Gamma)^{\lfloor \frac{q}{m} \rfloor - l - 1}(\widetilde{Q}_l + N_c)(H\Gamma^{\lfloor \frac{q}{m} \rfloor - l - 1})^{tr}.$$

At time instant tq, $t = 1, 2, \ldots$, we assume that $\hat{\mathbf{z}}_{\lfloor \frac{tq}{m} \rfloor|tq-1}$ and $E[(\mathbf{e}^z_{\lfloor \frac{tq}{m} \rfloor|tq-1})(\mathbf{e}^z_{\lfloor \frac{tq}{m} \rfloor|tq-1})^{tr}]$ are available at the transmitter. Since $E[(\mathbf{e}^z_{\lfloor \frac{tq}{m} \rfloor|tq-1})(\mathbf{e}^z_{\lfloor \frac{tq}{m} \rfloor|tq-1})^{tr}]$ is a Hermitian positive semi-definite matrix, it is possible to decompose it into the product of an upper triangular matrix and its conjugate transpose, i.e., $E[(\mathbf{e}^z_{\lfloor \frac{tq}{m} \rfloor|tq-1})(\mathbf{e}^z_{\lfloor \frac{tq}{m} \rfloor|tq-1})^{tr}] = \Omega_t \Omega_t^{tr}$.
Then, we consider the new variable $\tilde{\mathbf{z}}$ such that:

$$\tilde{\mathbf{z}}_{\lfloor \frac{tq}{m} \rfloor} \triangleq \Omega_t^{-1}\mathbf{z}_{\lfloor \frac{tq}{m} \rfloor}$$

$$\mathbf{e}^{\tilde{z}}_{\lfloor \frac{tq}{m} \rfloor|tq-1} \triangleq \tilde{\mathbf{z}}_{\lfloor \frac{tq}{m} \rfloor} - \hat{\tilde{\mathbf{z}}}_{\lfloor \frac{tq}{m} \rfloor|tq-1} = \Omega_t^{-1}\mathbf{e}^z_{\lfloor \frac{tq}{m} \rfloor|tq-1},$$

where its covariance matrix is the identity matrix, i.e., $E\{(\mathbf{e}^{\tilde{z}}_{\lfloor \frac{tq}{m} \rfloor|tq-1})(\mathbf{e}^{\tilde{z}}_{\lfloor \frac{tq}{m} \rfloor|tq-1})^{tr}\} = I_d$.
Subsequently, the following communication scheme is used at $tq \le i \le (t+1)q - 1$. During these q intervals, we assign q_j TDM time intervals to the jth element of $\tilde{\mathbf{z}}_{\lfloor \frac{tq}{m} \rfloor}$, $j \in \{1, 2, \ldots, d\}$. Assume that $tq + \sum_{l=1}^{J-1} q_l \le i \le tq + (\sum_{l=1}^{J} q_l) - 1$, such that at this interval (the ith interval), the Jth element of $\tilde{\mathbf{z}}_{\lfloor \frac{tq}{m} \rfloor}$ is transmitted. The transmission and estimation process at this time interval is as follows:

- Given $\hat{\tilde{\mathbf{z}}}_{\lfloor \frac{tq}{m} \rfloor|i-1}$, transmitter sends $f_i = (\alpha^2 - 1)^{\frac{1}{2}} d(i)(\tilde{\mathbf{z}}^{(J)}_{\lfloor \frac{tq}{m} \rfloor} - \hat{\tilde{\mathbf{z}}}^{(J)}_{\lfloor \frac{tq}{m} \rfloor|i-1} - n_{f_{i-1}})$, where $d(i)$ is the multiplying factor described as:

$$d(i) = \frac{\sqrt{N_0/2}}{\sqrt{E[(\tilde{\mathbf{z}}^{(J)}_{\lfloor \frac{tq}{m} \rfloor} - \hat{\tilde{\mathbf{z}}}^{(J)}_{\lfloor \frac{tq}{m} \rfloor|i-1})^2] + N_f}}, \tag{34}$$

- Upon receiving $v_i = f_i + n_i$, the receiver updates the estimation as:

$$\begin{aligned} \hat{\tilde{\mathbf{z}}}^{(J)}_{\lfloor \frac{tq}{m} \rfloor|i} &= \hat{\tilde{\mathbf{z}}}^{(J)}_{\lfloor \frac{tq}{m} \rfloor|i-1} + d(i)^{-1}\frac{(\alpha^2-1)^{\frac{1}{2}}}{\alpha^2} v_i, \\ \hat{\tilde{\mathbf{z}}}^{(j)}_{\lfloor \frac{tq}{m} \rfloor|i} &= \hat{\tilde{\mathbf{z}}}^{(j)}_{\lfloor \frac{tq}{m} \rfloor|i-1} \quad for \;\; j \neq J, \end{aligned} \tag{35}$$

  and sends it to the transmitter. Note that at time $i = (t+1)q - 1$, the receiver sends $\hat{\mathbf{z}}_{\lfloor \frac{(t+1)q}{m} \rfloor|i}$ to the transmitter through the feedback link and the transmitter obtains the value of $E[(\mathbf{e}^z_{\lfloor \frac{(t+1)q}{m} \rfloor|(t+1)q-1})(\mathbf{e}^z_{\lfloor \frac{(t+1)q}{m} \rfloor|(t+1)q-1})^{tr}]$.

The recursion relations for q intervals is solved to obtain $E[(\mathbf{e}^{\tilde{z}}_{\lfloor\frac{tq}{m}\rfloor|(t+1)q-1})(\mathbf{e}^{\tilde{z}}_{\lfloor\frac{tq}{m}\rfloor|(t+1)q-1})^{tr}] = \mathcal{P}(1)$. Therefore, $E[(\mathbf{e}^{z}_{\lfloor\frac{tq}{m}\rfloor|(t+1)q-1})(\mathbf{e}^{z}_{\lfloor\frac{tq}{m}\rfloor|(t+1)q-1})^{tr}] = \Omega_t\mathcal{P}(1)\Omega_t^{tr}$, and subsequently:

$$\begin{aligned}
&E\{(\mathbf{e}^{z}_{\lfloor\frac{(t+1)q}{m}\rfloor|(t+1)q-1})\mathbf{e}^{z}_{\lfloor\frac{(t+1)q}{m}\rfloor|(t+1)q-1})^{tr}\} = \Omega_{t+1}\Omega_{t+1}^{tr}\\
&= (H\Gamma)^{\lfloor\frac{(t+1)q}{m}\rfloor-\lfloor\frac{tq}{m}\rfloor}\Omega_t\mathcal{P}(1)\Omega_t^{tr}((H\Gamma)^{\lfloor\frac{(t+1)q}{m}\rfloor-\lfloor\frac{tq}{m}\rfloor})^{tr}+\\
&\sum_{l=\lfloor\frac{tq}{m}\rfloor}^{\lfloor\frac{(t+1)q}{m}\rfloor-1} H\Gamma^{\lfloor\frac{(t+1)q}{m}\rfloor-l-1}(\widetilde{Q}_l+N_c)(H\Gamma^{\lfloor\frac{(t+1)q}{m}\rfloor-l-1})^{tr}.
\end{aligned} \tag{36}$$

The second term in (36) is a bounded, Hermitian positive definite matrix. Therefore, there is an upper triangular matrix $\bar{\Delta}$ such that $\{\sum_{l=\lfloor\frac{tq}{m}\rfloor}^{\lfloor\frac{(t+1)q}{m}\rfloor-1} H\Gamma^{\lfloor\frac{(t+1)q}{m}\rfloor-l-1}(\widetilde{Q}_l+N_c)(H\Gamma^{\lfloor\frac{(t+1)q}{m}\rfloor-l-1})^{tr}\}_{ij} \leq \{\bar{\Delta}\bar{\Delta}^{tr}\}_{ij}$ for every $i,j \in \{1,2,\ldots,d\}$. We introduce $\bar{\Omega}_{k+1}$ as:

$$\bar{\Omega}_{t+1} \triangleq (H\Gamma)^{\lfloor\frac{(t+1)q}{m}\rfloor-\lfloor\frac{tq}{m}\rfloor}\bar{\Omega}_t\mathcal{P}(1)^{\frac{1}{2}} + \bar{\Delta}, \quad \bar{\Omega}_1 = \Omega_1. \tag{37}$$

Now, if $|\lambda_j|^{\frac{q}{m}} < \alpha^{q_j}$ for every j or equivalently $(\ln(|\lambda_j|) < \frac{mq_j}{q}\ln(\alpha))$, then $\bar{\Omega}_t$ is bounded (see Appendix B). Hence, $E\{(\mathbf{e}^{z}_{\lfloor\frac{tq}{m}\rfloor|tq-1})^{tr}(\mathbf{e}^{z}_{\lfloor\frac{tq}{m}\rfloor|tq-1})\} \leq trace\{\bar{\Omega}_t\bar{\Omega}_t^{tr}\} < \infty$.
Therefore, under this condition and by choosing controller gain as $\mathcal{G}_k = -(H\Gamma)$ for system (30), $E[(\mathbf{z}_k)^{tr}(\mathbf{z}_k)]$ is also bounded.

So far, we have concluded that the sufficient condition for bounded asymptotic stability (or asymptotic stability in the special case of $N_f = 0$, $N_C = 0$, and $G = 0$) is $(\ln|\lambda_j| < \frac{mq_j}{q}\ln(\alpha))$, $j = \{1,2,\ldots,d\}$. We now calculate the average transmission power to relate the derived condition to the channel capacity. Using equation (12), the average power for transmission equals to:

$$P_{av} = \sum_{j=1}^{d}\frac{q_j}{q}(P_{av})_j = \sum_{j=1}^{d}\frac{q_j}{q}\frac{N_0}{2}(\alpha^2-1) = \frac{N_0}{2}(\alpha^2-1), \tag{38}$$

where $(P_{av})_j$ is the average power used for transmission of jth element. Substituting the value of α in (38) results in $P_{av} = P$. On the other hand, for stability purposes, it is enough to have $(\ln(|\lambda_j| < \frac{mq_j}{q}\ln(\alpha))$ for every $j \in \{1,2,\ldots,d\}$ or equivalently:

$$\sum_{j=1}^{d}\ln|\lambda_j| < m\sum_{j=1}^{d}\frac{q_j}{q}\ln\alpha = m\ln(\alpha) = \frac{m}{2}\ln(1+\frac{P}{N_0/2}). \tag{39}$$

Under this condition, we are able to choose q_j such that $(\ln|\lambda_j| < \frac{mq_j}{q}\ln(\alpha))$ for every j, $j \in \{1,2,\ldots,d\}$. Since stable eigenvalues have no effect on this condition, in general when there are some stable eigenvalues, $\sum_{\{j=1,|\lambda_j|>1\}}^{d}\ln|\lambda_j| < \frac{m}{2}\ln(1+\frac{P}{N_0/2}) = mC$ is the sufficient condition for asymptotic and bounded asymptotic stability of the system (1) in the mean square sense. This completes the proof of Theorem 6. ■

Corollary 1. *For extending the results of Theorem 3 to the vector case, we should follow the similar steps explained in the part 1 of Theorem 6 with only the difference that instead of using encoder/decoder of case (A), we should use the encoder/decoder of case B. It should be noted that for this case, the stability condition is in the sense of* $\lim_{k\to\infty} E[(\mathbf{x}_k)^{tr}(\mathbf{x}_k)|\zeta^{\lfloor mk\rfloor}] \overset{a.s}{=} 0$.

Proof: Let we define a diagonal matrix $\mathscr{M}(i)$ whose elements for $j \in \{1, \ldots, d\}$ are obtained as:

$$\{\mathscr{M}(i)\}_{jj} = \Bigg(\prod_{t=0}^{\lfloor \frac{i+1}{q} \rfloor -1} \prod_{l=\sum_{k=1}^{j-1} q_k}^{\sum_{k=1}^{j} q_k -1} \big(a(\zeta_{tq+l})\big)^{-2} \Bigg) \prod_{l=\sum_{k=1}^{j-1} b_k(i)}^{\sum_{k=1}^{j} b_k(i)-1} \big(a(\zeta_{\lfloor \frac{i+1}{q} \rfloor q+l})\big)^{-2}.$$

In fact, $\{\mathscr{M}(i)\}_{jj}$ is obtained as the product of the parameters $\big(a(\zeta_\ell)\big)^{-2}$ where ℓ belongs to the set of time intervals up to time instant i in which $\mathbf{z}_0^{(j)}$ is transmitted .
Based on the proof of Theorem 3, after q time intervals, each component of $\mathbf{e}^z_{0|q-1} - \mathscr{M}(q-1)\mathbf{z}_0$ is a linear function of the channel noise whose coefficient are dependent on CSI. Therefore, $\mathbf{e}^z_{0|q-1} - \mathscr{M}(q-1)\mathbf{z}_0$ has a conditional normal distribution $N(\underline{0}_d, \mathscr{S}(q-1))$ where $\mathscr{S}(i)$ is a diagonal matrix whose jth element is satisfied the inequality $\{\mathscr{S}(i)\}_{jj} \leq \{\mathscr{M}(i)\}_{jj} \frac{\big(\lfloor \frac{i+1}{q} \rfloor q_j + b_j(i)\big) P}{\rho^2}$. Therefore, for this case of control/communication system, the covariance of matrix $\mathbf{e}^z_{k|\lfloor mk \rfloor}$ with respect to $\zeta^{\lfloor mk \rfloor}$ is obtained as (31) if we define $D = \mathscr{M}(\lfloor mk \rfloor)E[\mathbf{z_0}\mathbf{z_0}^{tr}]\mathscr{M}(\lfloor mk \rfloor) + \mathscr{S}(\lfloor mk \rfloor)$. Hence, the system is stable in the sense of $\lim_{k\to\infty} trace(E[\mathbf{e}^z_{k|\lfloor mk \rfloor}(\mathbf{e}^z_{k|\lfloor mk \rfloor})^{tr}|\zeta^{\lfloor mk \rfloor}]) \overset{a.s}{=} 0$, if the matrices $(H\Gamma)^k\mathscr{M}(\lfloor mk \rfloor)$ and $(H\Gamma)^k\mathscr{S}(\lfloor mk \rfloor)^{\frac{1}{2}}$ converge to zero matrix almost surely. By computing the eigenvalues of both matrices and based on Appendix A, it is concluded that if for every $j \in \{1, \ldots, d\}$, we have $\lim_{k\to\infty} |\lambda_j|(\prod_{t=0}^{\lfloor \frac{\lfloor mk \rfloor+1}{q} \rfloor -1} \prod_{l=\sum_{k=1}^{j-1} q_k}^{\sum_{k=1}^{j} q_k -1} a(\zeta_{tq+l}))^{\frac{1}{k}} \overset{a.s}{<} 1$, the both matrices converge to zero almost surely. Based on the law of large number, this condition is equivalent to $|\lambda_j| < m(q_j/q)E[a(\zeta_i)] = m(q_j/q)C$ for $j \in \{1, \ldots, d\}$. As previously explained, under condition $\sum_{\{j=1, |\lambda_j|>1\}}^{d} \ln|\lambda_j| < mC$, there are coefficients q_j and $j \in \{1, \ldots, d\}$ such that the inequalities $|\lambda_j| < m(q_j/q)E[a(\zeta_i)] = m(q_j/q)C$ and $j \in \{1, \ldots, d\}$ are satisfied. Note that using this encoding scheme, we have, $(P_{av})_j \leq P$ which results in $P_{av} \leq P$. This completes the proof. ■

Remark 4. *In this paper, it is proved that* $\sum_{\{j=1, |\lambda_j|>1\}}^{d} \ln|\lambda_j| < mC$ *is the necessary and sufficient condition on the stability of dynamic systems in different scenarios. Therefore, using the proposed scheme, we are able to stabilize the system in the case of* $T_s < T_c$ *(*$m < 1$*) by increasing the channel capacity which is equivalent to increasing the average transmitted power. Hence, the channel bandwidth (channel symbol rate) alone is not a limiting factor in achieving the stability.*

5. Conclusion

In this paper, we presented the necessary and sufficient conditions for stability of discrete-time linear control noisy systems over discrete-time AWGN and AWGN fading channels, when plant sample rate and communication channel symbol rate are not necessarily the same. We also analyzed the stability of system in the presence of process noise, controller-actuator noisy link, and communication feedback noisy link. It was shown that the eigenvalues rate condition is still a tight bound for stability under these conditions. For future, we try to extend the results to nonlinear systems.

Appendix A. Zero convergence condition for $(H\Gamma)^k D^{\frac{1}{2}}$

Due to space limitation, without loss of generality, we show these properties for the case of $H\Gamma = \begin{bmatrix} |\lambda_1| & 1 \\ 0 & |\lambda_2| \end{bmatrix}$. We should calculate the elements of matrix $(H\Gamma)^k$:

$$(H\Gamma)^k = \begin{bmatrix} |\lambda_1|^k & \{(H\Gamma)^k\}_{12} \\ 0 & |\lambda_2|^k \end{bmatrix},$$

where $\{(H\Gamma)^k\}_{12}$, $k \geq 2$ is obtained as follows:

$$\{(H\Gamma)^k\}_{12} = |\lambda_1|^{k-1} + |\lambda_2|\{(H\Gamma)^{k-1}\}_{12} = \sum_{j=1}^{k} |\lambda_2|^{(k-j)}|\lambda_1|^{j-1} = |\lambda_2|^{k-1}\frac{1-(|\lambda_1|/|\lambda_2|)^k}{1-|\lambda_1|/|\lambda_2|}. \quad \text{(A.1)}$$

Therefore, as $|\lambda_1| < |\lambda_2|$, $\{(H\Gamma)^k\}_{12} = \bar{C}|\lambda_2|^{k-1}$ for large k, where $\bar{C}$ is a constant.
Now, we consider the $(H\Gamma)^k D^{\frac{1}{2}}$ matrix, where $D^{\frac{1}{2}}$ is a diagonal matrix such that $\{D^{\frac{1}{2}}\}_{jj} = \frac{N_0}{2}\frac{1}{\alpha^{\lfloor(\lfloor mk\rfloor+1)/q\rfloor q_j + b_j(\lfloor mk\rfloor)}}$, $j = 1, 2$. Hence, for large k, we have:

$$(H\Gamma)^k D^{\frac{1}{2}} = \frac{N_0}{2}\begin{bmatrix} \frac{|\lambda_1|^{(k)}}{\alpha^{\lfloor(\lfloor mk\rfloor+1)/q\rfloor q_1+b_1(\lfloor mk\rfloor)}} & \frac{|\lambda_2|^{k-1}\bar{C}}{\alpha^{\lfloor(\lfloor mk\rfloor+1)/q\rfloor q_2+b_2(\lfloor mk\rfloor)}} \\ 0 & \frac{|\lambda_2|^{(k)}}{\alpha^{\lfloor(\lfloor mk\rfloor+1)/q\rfloor q_2+b_2(\lfloor mk\rfloor)}} \end{bmatrix}.$$

As a result, if $\frac{N_0}{2}\frac{|\lambda_j|^{(k)}}{\alpha^{\lfloor(\lfloor mk\rfloor+1)/q\rfloor q_j+b_j(\lfloor mk\rfloor)}}$, which is the eigenvalue of the matrix $(H\Gamma)^k D^{\frac{1}{2}}$, goes to zero for $j = 1, 2$, then the matrix $(H\Gamma)^k D^{\frac{1}{2}}$ will go to the zero matrix.
Note that if $|\lambda_1| = |\lambda_2|$, then $\{(H\Gamma)^k\}_{12} = k|\lambda_2|^{k-1}$, which leads to the similar results.

Appendix B. Bounded condition for $\bar{\Omega}_t$

Assume $\bar{\Omega}_{t+1} \triangleq (H\Gamma)^{\lfloor\frac{(t+1)q}{m}\rfloor-\lfloor\frac{tq}{m}\rfloor}\bar{\Omega}_t\mathcal{P}(1)^{\frac{1}{2}} + \bar{\Delta}$. By recursion relations we have:

$$\bar{\Omega}_t = (H\Gamma)^{\lfloor\frac{tq}{m}\rfloor-\lfloor\frac{q}{m}\rfloor}\Omega_1\mathcal{P}(1)^{\frac{t-1}{2}} + \sum_{l=1}^{t-1}(H\Gamma)^{\lfloor\frac{tq}{m}\rfloor-\lfloor\frac{(l+1)q}{m}\rfloor}\bar{\Delta}\mathcal{P}(1)^{\frac{t-1-l}{2}}, \quad \text{(B.1)}$$

We outline the proof of this property for the case in Appendix A. Note that $(H\Gamma)^{\lfloor\frac{tq}{m}\rfloor-\lfloor\frac{q}{m}\rfloor}\Omega_1\mathcal{P}(1)^{\frac{t-1}{2}}$ equals to the following matrix for large t:

$$\begin{bmatrix} \{\Omega_1\}_{11}|\lambda_1|^{r(0)}\{\mathcal{P}(1)\}_{11}^{\frac{t-1}{2}} & (\{\Omega_1\}_{12}|\lambda_1|^{r(0)}+\{\Omega_1\}_{22}\bar{C}|\lambda_2|^{r(0)-1})\{\mathcal{P}(1)\}_{22}^{\frac{t-1}{2}} \\ 0 & \{\Omega_1\}_{22}|\lambda_2|^{r(0)}\{\mathcal{P}(1)\}_{22}^{\frac{t-1}{2}} \end{bmatrix},$$

where $r(l) \triangleq \lfloor\frac{tq}{m}\rfloor - \lfloor\frac{(l+1)q}{m}\rfloor$. On the other hand, the second term in (B.1) is obtained as follows:

$$\begin{bmatrix} \sum_{l=1}^{t-1}\{\bar{\Delta}\}_{11}|\lambda_1|^{r(l)}\{\mathcal{P}(1)\}_{11}^{\frac{t-1-l}{2}} & \sum_{l=1}^{t-1}(\{\bar{\Delta}\}_{12}|\lambda_1|^{r(l)}+\{\bar{\Delta}\}_{22}\bar{C}|\lambda_2|^{r(l)-1})\{\mathcal{P}(1)\}_{22}^{\frac{t-1-l}{2}} \\ 0 & \sum_{l=1}^{t-1}\{\bar{\Delta}\}_{22}|\lambda_2|^{r(l)}\{\mathcal{P}(1)\}_{22}^{\frac{t-1-l}{2}} \end{bmatrix}.$$

Note that $\{\mathcal{P}(1)\}_{jj} = \frac{1}{\alpha^{2q_j}} + \sum_{\ell=0}^{q_j-2}(\frac{1}{\alpha^2})^{q_j-2-\ell}(1-\frac{1}{\alpha^2})N_f$, $r(l) = \lfloor\frac{tq}{m}\rfloor - \lfloor\frac{(l+1)q}{m}\rfloor$, and $|\lambda_1| < |\lambda_2|$, therefore, it is easy to verify that if $|\lambda_j|^{\frac{q}{m}} < \alpha^{q_j}$ or equivalently $\ln(|\lambda_j|) < m\frac{q_j}{q}\ln(\alpha)$ (for $j = 1, 2$), both the first and the second terms in (B.1) are bounded. Hence, $\bar{\Omega}_t$ is asymptotically bounded.

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