Stability of Linear Dynamic Systems over the Packet Erasure Channel: A Co-design Approach

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Abstract

This paper is concerned with the stability of linear time invariant dynamic systems over the packet erasure channel subject to minimum bit rate constraint when encoder and decoder are unaware of control signal. This assumption results in co-designing encoder, decoder and controller. Encoder, decoder, controller and conditions relating transmission rate to packet erasure probability and eigenvalues of the system matrix A are presented for almost sure asymptotic stability of linear time invariant dynamic systems over the packet erasure channel with feedback acknowledgment. When the eigenvalues of the system matrix A are real valued, it is shown that the obtained condition for stability is tight. Simulation result illustrates the satisfactory performance of the proposed encoder, decoder and controller for almost sure asymptotic stability.

Index Terms

Networked control system, almost sure stability, packet erasure channel.

I. Introduction

A. Motivation and Background

One of the issues that has begun to emerge in a number of applications is how to stabilize a dynamic system over a communication channel subject to imperfections (e.g., packet dropout, limited bit rate). Some examples of systems that are required to be stabilized over communication channels subject to imperfections are automated oil drilling system, smart oil well and coordination system of autonomous vehicles. One scenario for the latter example is the problem

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of coordination of an autonomous road vehicle using a set of cameras installed along the road. This cameras provide information on the position and orientation of autonomous vehicle with respect to the road and other vehicles for the on board controller of autonomous vehicle. Using the received information from the road, the on board controller is able to properly coordinates autonomous vehicle. Similar scenario is the coordination of unmanned aerial vehicles using off board sensors. In these scenarios, the information from cameras/sensors, which are not co-located with the controlled system, must be transmitted to the on board controller via (obviously) wireless links, in which this type of transmission is in general subject to imperfections.

Some results addressing basic problems in the stability of dynamic systems over communication channels subject to imperfections can be found in [1]-[13]. In the literature (e.g., [11]), under the assumptions that there exist noiseless feedback acknowledgments from receiver to encoder; and encoder and decoder are aware of control signal, a controller and a differential coding technique are presented for almost sure asymptotic stability of linear dynamic systems over the packet erasure channel with erasure probability α. It is also shown that under the above assumptions, the following condition

$$R_i > \frac{1}{1-\alpha}\max\{0, \log_2|\lambda_i(A)|\}, \quad i = 1, 2, ..., n, \tag{1}$$

where R_i bits is a component of the transmission rate $R = R_1 + R_2 + ... + R_n$ that corresponds to the ith eigenvalue $\lambda_i(A)$ of the system matrix $A \in \mathbb{R}^{n\times n}$, is a sufficient condition for almost sure asymptotic stability. That is, the stability of linear time invariant dynamic systems over the packet erasure channel with feedback acknowledgment is possible by implementing the rates R_is that satisfy the condition (1) in the coding technique of [11]. Furthermore, using data processing inequality [14], independent of the choice of encoder, decoder and controller, it is shown that $(1-\alpha)R \geq \sum_{i;|\lambda_i(A)|>1} \log_2|\lambda_i(A)|$, known as the eigenvalues rate condition, is a necessary condition for almost sure asymptotic stability of linear dynamic systems over the packet erasure channel. Obviously, there is a small gap between necessary condition and sufficient condition for almost sure asymptotic stability because the eigenvalues rate condition does not imply the stronger condition (1). For illustration, suppose $A = \begin{pmatrix} 5 & -1 \\ 6 & 0 \end{pmatrix}$ and $\alpha = 0.3$. Then, from the eigenvalues rate condition we have $R \geq 3.69$ and hence the smallest acceptable rate is $R = 4$ bits. But, from the stronger condition (1), we have $R_1 > 2.26$ and $R_2 > 1.42$ and hence the smallest rates for stability are $R_1 = 3$ bits and $R_2 = 2$ bits giving $R = 5$ bits as the smallest transmission rate for stability. Nevertheless, for those cases that the eigenvalues rate condition implies the stronger condition (1), we can conclude that the eigenvalues rate condition is a necessary and sufficient condition for almost sure asymptotic stability under the above assumptions; and hence, the condition (1) determines the minimum required transmission rate for stability. Because for many cases, the eigenvalues rate condition implies the stronger condition (1), throughout we are concerned with the cases where the eigenvalues rate condition and the condition (1) are equivalent.

For linear systems, the availability of control signal for both encoder and decoder simplifies the design of communication part (encoder and decoder) and controller because under this assumption the design of communication part and controller can be done separately [11]-[13]; instead of co-designing encoder, decoder and controller, which is obviously more difficult to do. But, in the discussed scenarios (the coordination of autonomous vehicles) encoder does not necessarily have access to control signal since sensors (and hence encoder) are not co-located with the controlled system which is directly affected by control signal. This motivates us to address a basic problem that can be associated with the discussed sensations where encoder and decoder are unaware of control signal. Moreover, the class of linear dynamic systems is an important class of systems. Many systems have linear dynamics. Also, nonlinear smooth dynamics, which includes a large number of real world nonlinear systems, can be approximated around working points by linear dynamics. The packet erasure channel with feedback acknowledgment is also an abstract model for the commonly used wireless information technologies, such as mobile communication, WiFi and Zigbi. The latter technology is subject to limited power consumption; and hence, its transmission is subject to limited bit rate constraint. Limitation in the power consumption of Zigbi modules also results in limitation in computational power of Zigbi module; because computation significantly consumes power; and hence, the implemented coding algorithm in Zigbi module must be as simple as possible to prolong the life time of Zigbi module. In addition, one of the desired stability criteria for the stability of dynamic systems is almost sure stability criterion. These motivate us to address the problem of almost sure stability of linear dynamic systems over the packet erasure channel with feedback acknowledgment when encoder and decoder are unaware of control signal, as shown in the block diagram of Fig. 1.

B. Paper Contributions

This paper extends the available results in almost sure asymptotic stability of linear dynamic systems over the packet erasure channel subject to minimum bit rate constraint by relaxing the assumption of the availability of control signal for encoder and decoder. Under these assumptions, an encoder, decoder and a controller are co-designed for linear dynamic systems over the packet

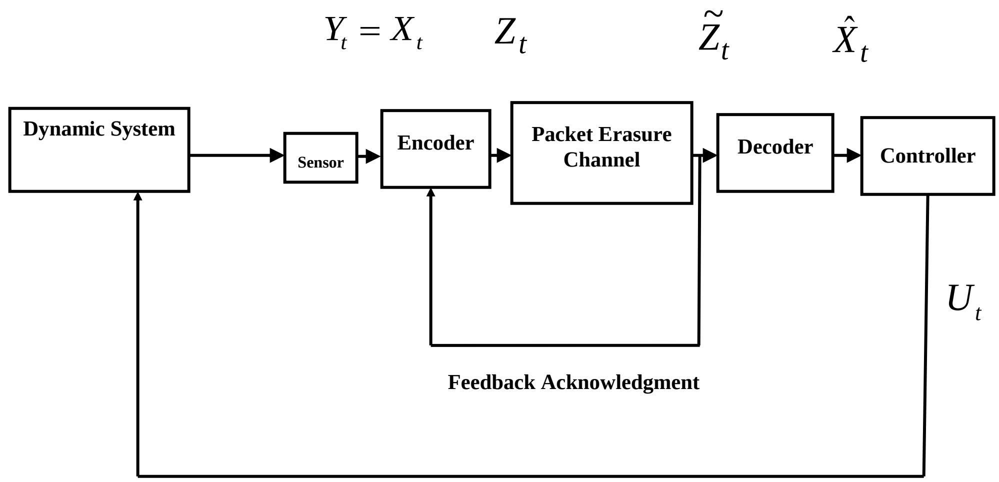

Fig. 1. A basic problem in control over communication.

erasure channel with feedback acknowledgment that results in almost sure asymptotic stability provided a condition relating transmission rate to packet erasure probability and eigenvalues of the system matrix A holds. When the eigenvalues of the system are real valued it is shown that the obtained condition is a necessary and sufficient condition for stability under the relaxed assumption; and hence, it is tight.

C. Paper Organization

The paper is organized as follows: In Section II the problem formulation is given. Section III is devoted to the stability result. In this section, an encoder, decoder, controller and a tight condition for almost sure asymptotic stability are presented. Simulation results are given in Section IV; and the paper is concluded in Section V by summarizing the main contributions of the paper.

II. Problem Formulation

Throughout, certain conventions are used: $|\cdot|$ denotes the absolute value, $\log_2(\cdot)$ denotes the logarithm of base 2 and '$\doteq$' means 'by definition is equivalent to'. $E[\cdot]$ denotes the expected value, $\mathbb{R}$ the set of real numbers and $diag(\cdot)$ denotes the diagonal/block diagonal matrix. $\tilde{Z}^t \doteq (\tilde{Z}_0, \tilde{Z}_1, ..., \tilde{Z}_t)$. $||\cdot||$ denotes the Euclidean norm and I_n the identity matrix with dimension n.

This paper is concerned with a basic problem in the stability of dynamic systems over communication channels subject to imperfections, as shown in the block diagram of Fig. 1.

The building blocks of Fig. 1 are described below.

Dynamic System: The dynamic system is described by the following fully observed linear time invariant system:

$$X_{t+1} = AX_t + BU_t, \quad X_0 = \xi, \quad Y_t = X_t, \quad t \in \mathbb{N}_+ \doteq \{0, 1, 2, 3, ...\}, \tag{2}$$

where $X_t \in \mathbb{R}^n$ is the state of the system, Y_t is the observation signal, the random variable ξ is the value of initial state, which is unknown for controller, and $U_t \in \mathbb{R}^m$ is the control signal. Throughout, it is assumed that the probability measure associated with the initial state X_0 with components $X_0^{(i)}$, $i = \{1, 2, ..., n\}$, has bounded support. That is, for each $i \in \{1, 2, ..., n\}$ there exists a compact set $[-L_0^{(i)}, L_0^{(i)}] \in \mathbb{R}$ such that $\Pr(X_0^{(i)} \in [-L_0^{(i)}, L_0^{(i)}]) = 1$. It is also assumed that the pair (A, B) is controllable.

Communication Channel: Communication channel between system and controller is the packet erasure channel with feedback acknowledgment. It is a digital channel that transmits a packet of binary data in each channel use. Let Z_t denote the channel input at time instant $t \in \mathbb{N}_+$, which is a packet of binary data containing information bits. Let also $\tilde{Z}_t$ denote the corresponding channel output. Also, let e denote the erasure symbol. Then,

$$\tilde{Z}_t = \begin{cases} Z_t & \text{with probability } 1-\alpha \\ e & \text{with probability } \alpha \end{cases}$$

That is, this channel erases a transmitted packet with probability α. Throughout, it is assumed that the erasure probability α is known a priori.

In the channel considered in this paper, there are feedback acknowledgments from receiver to encoder. That is, if a transmission is successful, an acknowledgment bit is sent from receiver to encoder indicating that the transmission was successful. The packet erasure channel with feedback acknowledgment is an abstract model for the commonly used data transmission technologies, such as the Internet, WiFi, mobile communication and Zigbi.

In the closed loop feedback system of Fig. 1, encoder and decoder are used to compensate the effects of random packet dropout.

Encoder: Encoder is a causal operator denoted by $Z_t = \mathcal{E}(Y_t, \tilde{Z}^{t-1})$ that maps the system output Y_t (by the knowledge of channel outputs) to channel input Z_t, which is a string of binaries with length R.

Decoder: Decoder is a causal operator denoted by $\hat{X}_t = \mathcal{D}(\tilde{Z}^t)$ that maps the channel output to $\hat{X}_t$ (the estimate of the state variable at decoder).

Controller: Controller has the following structure $U_t = K\hat{X}_t$, where K is the static controller gain.

The objective of this paper is to design an encoder, decoder and a controller that result in almost sure asymptotic stability of the system (2) defined in the following, by transmitting the minimum required bits.

Definition 2.1: (Almost Sure Asymptotic stability). Consider the block diagram of Fig. 1 described by the linear dynamic system (2) over the packet erasure channel, as described above. It is said that the system is almost sure asymptotic stabilizable if there exist an encoder, decoder and a controller such that the following property holds:

$$\Pr(\lim_{t\to\infty} X_t = 0) = 1.$$

III. Stability Result

In this section, we first address the stability question for the scalar system and then we extend the results to the vector case.

A. Scalar Case

In this section, we suppose that the dynamic system (2) is scalar and we present an encoder, decoder, controller and a necessary and sufficient condition on the length of transmitted packets R at each time instant, under which the dynamic system (2) in the block diagram of Fig. 1 is almost sure asymptotic stable.

Encoding and Decoding Technique: At time instant $t = 0$, we notice that $X_0 \in [-L_0, L_0]$. At this time instant, the encoder partitions the interval $[-L_0, L_0]$ into 2^R equal sized, non-overlapping sub-intervals and the center of each sub-interval is chosen as the index of that interval. Upon observing the initial state X_0, the index of the sub-interval that includes X_0 is encoded into R bits and transmitted to the decoder through the packet erasure channel. If the decoder receives this R bits successfully, it identifies the index of the sub-interval where X_0 lives in; and the value of this index is chosen as $\hat{X}_0$ (the estimate of X_0 at receiver). Therefore, the estimation error for this case is bounded above by $|X_0 - \hat{X}_0| \leq V_0 = \frac{L_0}{2^R}$. But if erasure occurs, then $\hat{X}_0 = 0$; and therefore, $|X_0 - \hat{X}_0| \leq V_0 = L_0$.

At time instant $t = 1$, $X_1 = AX_0 + BK\hat{X}_0$. Let $E_0 \doteq X_0 - \hat{X}_0$, then $X_1 = (A+BK)X_0 - BKE_0 \in [-L_1, L_1]$, where

$$L_1 = |A + BK|L_0 + |BK|M_0L_0, \quad M_0 = \begin{cases} \frac{1}{2^R}, & \Pr(M_0 = \frac{1}{2^R}) = 1 - \alpha \\ 1, & \Pr(M_0 = 1) = \alpha. \end{cases} \tag{3}$$

Note that M_0 is the indicator of successful transmission or failed transmission at time instant $t = 0$. As the encoder has access to the feedback acknowledgment of the time instant $t = 0$, it knows the value of M_0 at time instant $t = 1$.
Similar to the previous time instant, at time instant $t = 1$, the encoder partitions the interval $[-L_1, L_1]$ into 2^R equal sized, non-overlapping sub-intervals and the center of each sub-interval is chosen as the index of that interval. Upon observing X_1, the index of the sub-interval that includes X_1 is encoded into R bits and transmitted to the decoder through the packet erasure channel. Similarly, if these R bits are received successfully, the decoder identifies the index of the sub-interval that contains X_1 and the value of this index is chosen as $\hat{X}_1$. Therefore, the estimation error for this case is bounded above by $|E_1| \doteq |X_1 - \hat{X}_1| \leq V_1 = \frac{L_1}{2^R}$. But if erasure occurs, then $\hat{X}_1 = 0$; and therefore $|E_1| = |X_1 - \hat{X}_1| \leq V_1 = L_1$.
By following a similar procedure, as described above, the sequence $\hat{X}_0$, $\hat{X}_1$, $\hat{X}_2$, ... are constructed at the decoder.

Now, we must show that using the above coding technique there exists a controller that results in almost sure asymptotic stability. This result is shown in the following proposition.

Proposition 3.1: Consider the control system of Fig. 1 described by the dynamic system (2) over the packet erasure channel with erasure probability α, as described earlier. Suppose that the transmission rate R satisfies the following inequality:

$$(1 - \alpha)R > \max\{0, \log_2 |A|\}. \tag{4}$$

Then, using the proposed encoding and decoding technique and $U_t = -\frac{A}{B}\hat{X}_t$, we have almost sure asymptotic stability of the form $X_t \to 0$, P-a.s.; or equivalently, $\Pr(\lim_{t\to\infty} X_t = 0) = 1$.
Proof: Choose any rate R that satisfies the condition (4). For this rate, define the random variable M_t as follows:

$$M_t = \begin{cases} \frac{1}{2^R}, & \Pr(M_t = \frac{1}{2^R}) = 1 - \alpha \\ 1, & \Pr(M_t = 1) = \alpha. \end{cases} \tag{5}$$

This random variable is the indicator of successful transmission or failed transmission at time instant t. Therefore, it is independent of the other variables $M_{t'}$, $t'(\in \mathbb{N}_+) \neq t$. It is also clear from (5) that the process $\{M_t\}_{t\in\mathbb{N}_+}$ is identically distributed. So, the random process $\{M_t\}_{t\in\mathbb{N}_+}$ is an i.i.d. process.

Using the above encoding and decoding technique and controller $U_t = K\hat{X}_t$, we have

$$X_{t+1} = (A + BK)X_t - BKE_t, \quad (E_t \doteq X_t - \hat{X}_t), \quad X_0 \in [-L_0, L_0],$$

and hence

$$
\begin{aligned}
|X_1| &\le |A+BK|L_0 + |BK|M_0L_0 \doteq L_1\\
|X_2| &\le |A+BK|L_1 + |BK|M_1L_1 \doteq L_2\\
&.\\
&.\\
&.\\
|X_t| &\le |A+BK|L_{t-1} + |BK|M_{t-1}L_{t-1} \doteq L_t.
\end{aligned}
$$

From the recursive equation $L_t = (|A+BK| + |BK|M_{t-1})L_{t-1}$, which defines an upper bound on $|X_t|$, it follows that

$$
L_t = L_0 \prod_{j=0}^{t-1}(|A+BK| + |BK|M_j) = L_0 2^{t(\frac{1}{t}\sum_{j=0}^{t-1}\log_2(|A+BK|+|BK|M_j))}. \tag{6}
$$

From the definition that we have for the i.i.d. process M_j, for the i.i.d. process $|A+BK| + |BK|M_j$, we also have

$$
|A+BK| + |BK|M_j = \begin{cases} |A+BK| + \frac{|BK|}{2^R}, & \text{with probability } 1-\alpha \\ |A+BK| + |BK|, & \text{with probability } \alpha. \end{cases}
$$

Therefore, from the strong law of large numbers [15], we have

$$
\lim_{t\to\infty}\frac{1}{t}\sum_{j=0}^{t-1}\log_2(|A+BK| + |BK|M_j) = E[\log_2(|A+BK| + |BK|M_0)]
$$

$$
= (1-\alpha)\log_2(|A+BK| + \frac{|BK|}{2^R}) + \alpha\log_2(|A+BK| + |BK|). \tag{7}
$$

Now, if the rate R is chosen such that the following inequality holds

$$
(1-\alpha)\log_2(|A+BK| + \frac{|BK|}{2^R}) + \alpha\log_2(|A+BK| + |BK|) < 0; \tag{8}
$$

then from (7) and (6) it follows that $L_t \to 0$, as $t \to \infty$, P-a.s.; and hence, $X_t \to 0$, P-a.s. For the stabilizing gain $K = -\frac{A}{B}$, the condition (8) is reduced to the condition (4). Hence, using the proposed encoding and decoding technique with rate R satisfying the condition (4) and controller $U_t = -\frac{A}{B}\hat{X}_t$, the system is almost sure asymptotically stable.

We have the following corollary as a result of the above proposition and ([11], Proposition 3.3).

Corollary 3.2: The condition (4) is a necessary and sufficient condition for almost sure asymptotic stability of the scalar version of the dynamic system (2) over the packet erasure channel as shown in the block diagram of Fig. 1.

Proof: From the above proposition it follows that the condition (4) is a sufficient condition under which there exist an encoder, decoder and a controller for almost sure asymptotic stability of the scalar version of the dynamic system (2) over the packet erasure channel. On the other hand, as shown in ([11], Proposition 3.3), independent of the choice of encoder, decoder and controller, the condition (4) is a necessary condition for almost sure asymptotic stability of linear scalar dynamic systems over the packet erasure channel. Hence, the condition (4) is a necessary and sufficient condition.

Remark 3.3: Using the proposed encoding and decoding technique and the controller $U_t = -\frac{A}{B}\hat{X}_t$ and by setting the transmission rate R as the smallest integer that is greater than or equivalent to $\frac{1}{1-\alpha}\max\{0, \log_2|\lambda_i(A)|\}$, almost sure asymptotic stability by transmitting the minimum required bits is achieved.

As the encoder and decoder are unaware of control signal, unlike [11], to have almost sure asymptotic stability by transmitting with the minimum required bits, the design of communication part (encoder and decoder) and controller cannot be done separately; and encoder, decoder and controller must be co-designed. Under the assumption that encoder and decoder are aware of control signal, in the design of communication part, without loss of generality, we can assume that $U_t = 0$ as encoder by the knowledge of control signal can exclude the effects of control signal from encoded message; and subsequently, decoder by including the effects of control signal to the reconstructed message, reconstructs the state variable of the controlled system. Using this approach, a differential coding technique, which transmits the quantized version of the error between message and an estimate of the reconstructed message at decoder, is presented in [11] for almost sure asymptotic tracking of the form $\hat{X}_t \to X_t$, P-a.s., with minimum transmission rate. On the other hand, under the separation design approach, in the design of controller, we can assume $\hat{X}_t = X_t$. That is, under the separation design approach, controller is designed without considering communication imperfections; and consequently, any gain K such that $|A+BK| < 1$ results in almost sure asymptotic stability by transmitting with minimum required bits. However, under the co-design approach, as it is clear from the inequality (8), the design of controller affects the design of communication part such that the static controller gain K must be chosen as $K = -\frac{A}{B}$ to have stability by transmitting with the minimum required bits using controller $U_t = K\hat{X}_t$ and the proposed encoding and decoding technique. For partially observed continuous and discrete time linear dynamic systems over Additive White Gaussian Noise (AWGN) channel, under the assumption that encoder and decoder are aware of control signal, the separation design approach also results in desired tracking and stability [12],[13].

B. Vector Case

Now, suppose $X_t \in \mathbb{R}^n$, the system matrix A has distinct real eigenvalues, real multiple eigenvalues and distinct complex conjugate eigenvalues. Suppose also that the matrix B is invertible. Hence, there always exists an invertible real similarity transformation matrix $\Phi \in \mathbb{R}^{n\times n}$ such that $\Phi A \Phi^{-1} \doteq \Gamma = diag(J_1, J_2, ..., J_m)$ [16]. Where each J_i, $i = 1, 2, ..., m$, is a Jordan block. The Jordan block associated with a real eigenvalue $\lambda_i(A)$ with multiplicity d_i is

$$\begin{pmatrix} \lambda_i(A) & a & 0 & . & . & . & 0 & 0 \\ 0 & \lambda_i(A) & b & . & . & . & 0 & 0 \\ . & & & & & & & \\ . & & & & & & & \\ . & & & & & & & \\ 0 & 0 & 0 & . & . & . & \lambda_i(A) & c \\ 0 & 0 & 0 & . & . & . & 0 & \lambda_i(A) \end{pmatrix}$$

where $a, b, c \in \{0, 1\}$ depending on the rank of matrix $(\lambda_i(A)I_n - A)$. The Jordan block associated with the complex conjugate pair of eigenvalues $\lambda_i(A) = \sigma \pm \sqrt{-1}\omega$ $(\omega \neq 0)$ is $\begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}$.

Now, consider the following similarity transformation $Z_t = \Phi X_t$. Under this transformation, the system (2) is transformed to the following system

$$Z_{t+1} = \Gamma Z_t + \Phi B U_t. \tag{9}$$

Let $U_t = (\Phi B)^{-1} V_t$, where $V_t = (\, V_t^{(1)} \quad V_t^{(2)} \quad . \quad . \quad . \quad V_t^{(n)} \,)' \in \mathbb{R}^n$ is the control vector of the transformed system (9), which has the following equivalent representation:

$$Z_{t+1} = \Gamma Z_t + V_t. \tag{10}$$

System (10) consists of m independent sub-systems of the following form $Z_{t+1}^{(i)} = J_i Z_t^{(i)} + V_t^{(i)}$, $i = \{1, 2, ..., m\}$. Therefore, to show that the vector Z_t converges to zero, almost surely, it is enough to show that each independent sub-system is asymptotically stable, almost surely.

Note that as $Z_t = \Phi X_t$, where Φ is an invertible matrix, almost sure asymptotic stability of Z_t implies almost sure asymptotic stability of X_t and vice versa. Hence, by addressing the almost sure stability question of Z_t, we address the stability question of X_t. Note also that at time instant $t = 0$, using the transformation ΦX_0 the encoder and decoder can compute the boxes

$[-H_0^{(j)}, H_0^{(j)}]$, $j = \{1, 2, ..., n\}$ that contain the components of the vector Z_0.

For the vector case, at each time instant t, upon observing X_t, the encoder computes $Z_t = \Phi X_t$ and the encoding and decoding technique of Section III-A is applied to each component of the vector Z_t by defining upper bounds on each component of the vector Z_t. When the encoder encodes all the components of the vector Z_t, it transmits a packet with length $R = R_1 + R_2 + ... + R_n$ bits to the decoder through the packet erasure channel; and the decoder reconstructs each component of the vector Z_t following the decoding technique of Section III-A and outputs $\hat{Z}_t$, which is the reconstruction of Z_t at the end of communication.

Having that, in the following propositions, we design control vector V_t; and we present sufficient conditions on each component of the transmission rate to have almost sure asymptotic stability of Z_t; and hence, almost sure asymptotic stability of X_t by applying $U_t = (\Phi B)^{-1} V_t$.

We start from the case of $J_i = \lambda_i(A)$, where $\lambda_i(A)$ is a real distinct eigenvalue.

Proposition 3.4: Consider the control system of Fig. 1 described by the dynamic system (11) over the packet erasure channel with erasure probability α, as described earlier.

$$Z_{t+1}^{(j)} = J_i Z_t^{(j)} + V_t^{(j)}, \quad Z_t^{(j)}, V_t^{(j)} \in \mathbb{R}, \quad J_i = \lambda_i(A). \tag{11}$$

Suppose that the rate R_j satisfies the following inequality $(1-\alpha)R_j > \max\{0, \log_2 |\lambda_i(A)|\}$. Then, by implementing the control signal $V_t^{(j)} = -\lambda_i(A)\hat{Z}_t^{(j)}$, we have almost sure asymptotic stability of the form $Z_t^{(j)} \to 0$, P-a.s.

Proof: The proof follows along the same lines of the proof of Proposition 3.1.

Now, we extend the above result when J_i corresponds to real eigenvalues with multiplicity $d_i > 1$. In the following proposition, without loss of generality, we suppose that $J_i = \begin{pmatrix} \lambda_i(A) & 1 \\ 0 & \lambda_i(A) \end{pmatrix}$ and we design control vector and present a sufficient condition on transmission rates for almost sure asymptotic stability.

Proposition 3.5: Consider the control system of Fig. 1 described by the dynamic system (12) over the packet erasure channel with erasure probability α, as described earlier.

$$\begin{pmatrix} Z_{t+1}^{(j)} \\ Z_{t+1}^{(j+1)} \end{pmatrix} = \begin{pmatrix} \lambda_i(A) & 1 \\ 0 & \lambda_i(A) \end{pmatrix} \begin{pmatrix} Z_t^{(j)} \\ Z_t^{(j+1)} \end{pmatrix} + \begin{pmatrix} V_t^{(j)} \\ V_t^{(j+1)} \end{pmatrix}. \tag{12}$$

Suppose that the rate $R_j = R_{j+1}$ satisfy the following inequality

$$(1-\alpha)R_j > \max\{0, \log_2 |\lambda_i(A)|\}. \tag{13}$$

Then, by implementing the control signals $V_t^{(j)} = -\lambda_i(A)\hat{Z}_t^{(j)} - \hat{Z}_t^{(j+1)}$ and $V_t^{(j+1)} = -\lambda_i(A)\hat{Z}_t^{(j+1)}$, we have almost sure asymptotic stability of the form $Z_t^{(j)} \to 0$, P-a.s. and $Z_t^{(j+1)} \to 0$, P-a.s.

Proof: From the dynamic system (12) for $V_t^{(j)} = -\lambda_i(A)\hat{Z}_t^{(j)} - \hat{Z}_t^{(j+1)}$ and $V_t^{(j+1)} = -\lambda_i(A)\hat{Z}_t^{(j+1)}$ it follows that $Z_{t+1}^{(j+1)} = \lambda_i(A)Z_t^{(j+1)} - \lambda_i(A)\hat{Z}_t^{(j+1)} = \lambda_i(A)E_t^{(j+1)}$, where $E_t^{(j+1)} \doteq Z_t^{(j+1)} - \hat{Z}_t^{(j+1)}$. Now, for this dynamic system along the same lines of the proof of Proposition 3.1, it is shown that $Z_t^{(j+1)} \to 0$, P-a.s. as $R_{j+1} = R_j > \frac{1}{1-\alpha}\max\{0, \log_2|\lambda_i(A)|\}$.

For the dynamic system (12) for $Z_t^{(j)}$ we have

$$\begin{aligned} Z_{t+1}^{(j)} &= \lambda_i(A)Z_t^{(j)} + Z_t^{(j+1)} - \lambda_i(A)\hat{Z}_t^{(j)} - \hat{Z}_t^{(j+1)} \\ &= \lambda_i(A)E_t^{(j)} + E_t^{(j+1)}, \quad E_t^{(j)} \doteq Z_t^{(j)} - \hat{Z}_t^{(j)}. \end{aligned} \tag{14}$$

Now, let $L_t^{(j)}$ be an upper bound on $|Z_t^{(j)}|$, that is, $|Z_t^{(j)}| \leq L_t^{(j)}$. Then, from the dynamic system (14) $L_t^{(j)}$ is calculated as follows

$$\begin{aligned} |Z_0^{(j)}| &\leq H_0^{(j)} = L_0^{(j)} \\ |Z_1^{(j)}| &\leq |\lambda_i(A)||E_0^{(j)}| + |E_0^{(j+1)}| \leq |\lambda_i(A)|M_0H_0^{(j)} + M_0H_0^{(j+1)} = L_1^{(j)} \\ |Z_2^{(j)}| &\leq |\lambda_i(A)||E_1^{(j)}| + |E_1^{(j+1)}| \leq |\lambda_i(A)|M_1L_1^{(j)} + M_1L_1^{(j+1)} = L_2^{(j)} \\ &. \\ &. \\ &. \\ |Z_t^{(j)}| &\leq |\lambda_i(A)||E_{t-1}^{(j)}| + |E_{t-1}^{(j+1)}| \leq |\lambda_i(A)|M_{t-1}L_{t-1}^{(j)} + M_{t-1}L_{t-1}^{(j+1)} = L_t^{(j)}. \end{aligned}$$

Note that M_t is an i.i.d. stochastic process with the following description

$$M_t = \begin{cases} \frac{1}{2^{R_j}}, & \Pr(M_t = \frac{1}{2^{R_j}}) = 1 - \alpha \\ 1, & \Pr(M_t = 1) = \alpha. \end{cases}$$

From the above analysis it follows that

$$\begin{pmatrix} L_t^{(j)} \\ L_t^{(j+1)} \end{pmatrix} = M_{t-1}\begin{pmatrix} |\lambda_i(A)| & 1 \\ 0 & |\lambda_i(A)| \end{pmatrix}\begin{pmatrix} L_{t-1}^{(j)} \\ L_{t-1}^{(j+1)} \end{pmatrix}, \begin{pmatrix} L_0^{(j)} \\ L_0^{(j+1)} \end{pmatrix} = \begin{pmatrix} H_0^{(j)} \\ H_0^{(j+1)} \end{pmatrix}. \tag{15}$$

Now, by expanding the dynamic model (15) it follows that

$$\begin{pmatrix} L_t^{(j)} \\ L_t^{(j+1)} \end{pmatrix} = (\prod_{j=0}^{t-1} M_j)\begin{pmatrix} |\lambda_i(A)|^t & t|\lambda_i(A)|^{t-1} \\ 0 & |\lambda_i(A)|^t \end{pmatrix}\begin{pmatrix} H_0^{(j)} \\ H_0^{(j+1)} \end{pmatrix}. \tag{16}$$

Now, in the following, we show that all the components of the following matrix

$$\begin{pmatrix} (\prod_{j=0}^{t-1} M_j)|\lambda_i(A)|^t & (\prod_{j=0}^{t-1} M_j)t|\lambda_i(A)|^{t-1} \\ 0 & (\prod_{j=0}^{t-1} M_j)|\lambda_i(A)|^t \end{pmatrix} \tag{17}$$

converge to zero, P-a.s.

As the condition (13) holds, for the component $(\prod_{j=0}^{t-1} M_j)|\lambda_i(A)|^t$ along the same lines of the proof of Proposition 3.1from the strong law of large numbers, it follows that

$$\lim_{t\to\infty}(\prod_{j=0}^{t-1} M_j)|\lambda_i(A)|^t \to 0, \text{ P-a.s.}$$

For the other component, we have the following

$$\begin{aligned}(\prod_{j=0}^{t-1} M_j)(t|\lambda_i(A)|^{t-1}) &= t\prod_{j=0}^{t-1} M_j|\lambda_i(A)| = t2^{\log_2 \prod_{j=0}^{t-1} M_j|\lambda_i(A)|} \\ &= t2^{\sum_{j=0}^{t-1}\log_2(M_j|\lambda_i(A)|)} \\ &= t2^{t(\frac{1}{t}\sum_{j=0}^{t-1}\log_2 M_j|\lambda_i(A)|)}.\end{aligned}$$

Now, as the condition (13) holds, from the strong law of large numbers it follows that

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}\sum_{j=0}^{t-1}\log_2(M_j|\lambda_i(A)|) &= E[\log_2(M_j|\lambda_i(A)|)] = (1-\alpha)\log_2\frac{|\lambda_i(A)|}{2^{R_j}} + \alpha\log_2|\lambda_i(A)| \\ &= (1-\alpha)\log_2|\lambda_i(A)| - (1-\alpha)R_j + \alpha\log_2|\lambda_i(A)| \\ &= \log_2|\lambda_i(A)| - (1-\alpha)R_j \doteq \gamma < 0.\end{aligned}$$

Hence, by applying the rule of Hopital-Bernouli for limits it follows that

$$\lim_{t\to\infty}(\prod_{j=0}^{t-1} M_j)(t|\lambda_i(A)|^{t-1}) = \lim_{t\to\infty}\frac{t}{2^{-\gamma t}} = \lim_{t\to\infty}\frac{1}{(\ln 2)2^{-\gamma t}} = 0.$$

Hence, from the above analysis it follows that the matrix (17) converges to zero, P-a.s.; and therefore, from (16) it follows that $L_t^{(j)} \to 0$ and $L_t^{(j+1)} \to 0$, P-a.s. This completes the proof as $L_t^{(j)}$ and $L_t^{(j+1)}$ are upper bounds for $|Z_t^{(j)}|$ and $|Z_t^{(j+1)}|$, respectively.

We have the following corollary as a result of the above propositions.

Corollary 3.6: Consider the dynamic system (2) and suppose that the eigenvalues of the matrix A are real valued. Then, the obtained conditions in Propositions 3.4 and 3.5 together is a necessary and sufficient condition for almost sure asymptotic stability of the system (2) over the packet erasure channel as shown in the block diagram of Fig. 1.

Proof: From ([11], Proposition 3.3) it follows that this condition is a necessary condition for almost sure asymptotic stability. From this result combined with the result of Propositions 3.4 and 3.5 it follows that this condition is a necessary and sufficient condition for almost sure asymptotic stability. That is, it is tight.

Now, in the following proposition, we are concerned with the Jordan block J_i that corresponds to the complex conjugate pair of eigenvalues $\sigma + \sqrt{-1}\omega$, $\omega \neq 0$.

Proposition 3.7: Consider the control system of Fig. 1 described by the dynamic system (18) over the packet erasure channel with erasure probability α, as described earlier.

$$\begin{pmatrix} Z_{t+1}^{(j)} \\ Z_{t+1}^{(j+1)} \end{pmatrix} = \begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix} \begin{pmatrix} Z_t^{(j)} \\ Z_t^{(j+1)} \end{pmatrix} + \begin{pmatrix} V_t^{(j)} \\ V_t^{(j+1)} \end{pmatrix}. \tag{18}$$

Suppose that the rate $R_j = R_{j+1}$ satisfy the following inequality

$$(1-\alpha)R_j > \max\{0, \log_2(|\sigma| + |\omega|)\}. \tag{19}$$

Then, by implementing the control signals $V_t^{(j)} = -\sigma\hat{Z}_t^{(j)} - \omega\hat{Z}_t^{(j+1)}$ and $V_t^{(j+1)} = \omega\hat{Z}_t^{(j)} - \sigma\hat{Z}_t^{(j+1)}$, we have almost sure asymptotic stability of the form $Z_t^{(j)} \to 0$, P-a.s. and $Z_t^{(j+1)} \to 0$, P-a.s.

Proof: From the dynamic system (18) for $V_t^{(j)} = -\sigma\hat{Z}_t^{(j)} - \omega\hat{Z}_t^{(j+1)}$ and $V_t^{(j+1)} = \omega\hat{Z}_t^{(j)} - \sigma\hat{Z}_t^{(j+1)}$, it follows that

$$Z_t^{(j)} = \sigma E_t^{(j)} + \omega E_t^{(j+1)} \tag{20}$$

$$Z_t^{(j+1)} = -\omega E_t^{(j)} + \sigma E_t^{(j+1)}. \tag{21}$$

Now, from the dynamic systems (20) and (21) the upper bounds $L_t^{(j)}$ and $L_t^{(j+1)}$ on $|Z_t^{(j)}|$ and $|Z_t^{(j+1)}|$ are calculated as follows

$$\begin{aligned}
|Z_0^{(j)}| &\leq H_0^{(j)} = L_0^{(j)} \\
|Z_1^{(j)}| &\leq |\sigma||E_0^{(j)}| + |\omega||E_0^{(j+1)}| \leq |\sigma| M_0 H_0^{(j)} + |\omega| M_0 H_0^{(j+1)} = L_1^{(j)} \\
|Z_2^{(j)}| &\leq |\sigma||E_1^{(j)}| + |\omega||E_1^{(j+1)}| \leq |\sigma| M_1 L_1^{(j)} + |\omega| M_1 L_1^{(j+1)} = L_2^{(j)} \\
&. \\
&. \\
&. \\
|Z_t^{(j)}| &\leq |\sigma||E_{t-1}^{(j)}| + |\omega||E_{t-1}^{(j+1)}| \leq |\sigma| M_{t-1} L_{t-1}^{(j)} + |\omega| M_{t-1} L_{t-1}^{(j+1)} = L_t^{(j)}.
\end{aligned}$$

$$
\begin{aligned}
|Z_0^{(j+1)}| &\leq H_0^{(j+1)} = L_0^{(j+1)} \\
|Z_1^{(j+1)}| &\leq |\omega||E_0^{(j)}| + |\sigma||E_0^{(j+1)}| \leq |\omega| M_0 H_0^{(j)} + |\sigma| M_0 H_0^{(j+1)} = L_1^{(j+1)} \\
|Z_2^{(j+1)}| &\leq |\omega||E_1^{(j)}| + |\sigma||E_1^{(j+1)}| \leq |\omega| M_1 L_1^{(j)} + |\sigma| M_1 L_1^{(j+1)} = L_2^{(j+1)} \\
&\cdot \\
&\cdot \\
&\cdot \\
|Z_t^{(j+1)}| &\leq |\omega||E_{t-1}^{(j)}| + |\sigma||E_{t-1}^{(j+1)}| \leq |\omega| M_{t-1} L_{t-1}^{(j)} + |\sigma| M_{t-1} L_{t-1}^{(j+1)} = L_t^{(j+1)}.
\end{aligned}
$$

From the above analysis it follows that

$$
\begin{pmatrix} L_t^{(j)} \\ L_t^{(j+1)} \end{pmatrix} = M_{t-1} \begin{pmatrix} |\sigma| & |\omega| \\ |\omega| & |\sigma| \end{pmatrix} \begin{pmatrix} L_{t-1}^{(j)} \\ L_{t-1}^{(j+1)} \end{pmatrix}, \quad \begin{pmatrix} L_0^{(j)} \\ L_0^{(j+1)} \end{pmatrix} = \begin{pmatrix} H_0^{(j)} \\ H_0^{(j+1)} \end{pmatrix}. \tag{22}
$$

Now, as the matrix $\begin{pmatrix} |\sigma| & |\omega| \\ |\omega| & |\sigma| \end{pmatrix}$ has two real distinct eigenvalues $|\sigma| - |\omega|$ and $|\sigma| + |\omega|$, by applying a similarity transformation of the following form

$$
\begin{pmatrix} Q_t^{(j)} \\ Q_t^{(j+1)} \end{pmatrix} = \Phi \begin{pmatrix} L_t^{(j)} \\ L_t^{(j+1)} \end{pmatrix},
$$

the dynamic system (22) can be transformed to the following system

$$
\begin{pmatrix} Q_t^{(j)} \\ Q_t^{(j+1)} \end{pmatrix} = M_{t-1} \begin{pmatrix} |\sigma| - |\omega| & 0 \\ 0 & |\sigma| + |\omega| \end{pmatrix} \begin{pmatrix} Q_{t-1}^{(j)} \\ Q_{t-1}^{(j+1)} \end{pmatrix}.
$$

Now, along the same lines of the proof of Proposition 3.1 it follows that $Q_t^{(j)} \to 0$ and $Q_t^{(j+1)} \to 0$, P-a.s.; and hence, $L_t^{(j)}$ and $L_t^{(j+1)}$ converge to zero, P-a.s., as $(1-\alpha)R_j > \max\{0, \log_2(|\sigma| + |\omega|)\}(R_j = R_{j+1})$. This completes the proof.

Remark 3.8: From ([11], Proposition 3.3), under the assumption of $R_j = R_{j+1}$, it follows for the system (18) that $R_j = R_{j+1} \geq \frac{1}{1-\alpha} \max\{0, \log_2 \sqrt{\sigma^2 + \omega^2}\}$ is a necessary condition for almost sure asymptotic stability. From Proposition 3.7 it also follows that the condition $R_j = R_{j+1} > \max \frac{1}{1-\alpha}\{0, \log_2(|\sigma| + |\omega|)\}$ is a sufficient condition for almost sure asymptotic stability. Hence, there is a small gap for this case between necessary condition and sufficient condition; because $|\sigma| + |\omega| = \sqrt{|\sigma|^2 + |\omega|^2 + 2|\sigma||\omega|} \geq \sqrt{\sigma^2 + \omega^2}$.

Remark 3.9: If the linear dynamics is an approximation, that is, there is uncertainty in the linear dynamic model (2), then as it is shown in [17], almost sure asymptotic stability of the system over the packet erasure channel is not possible.

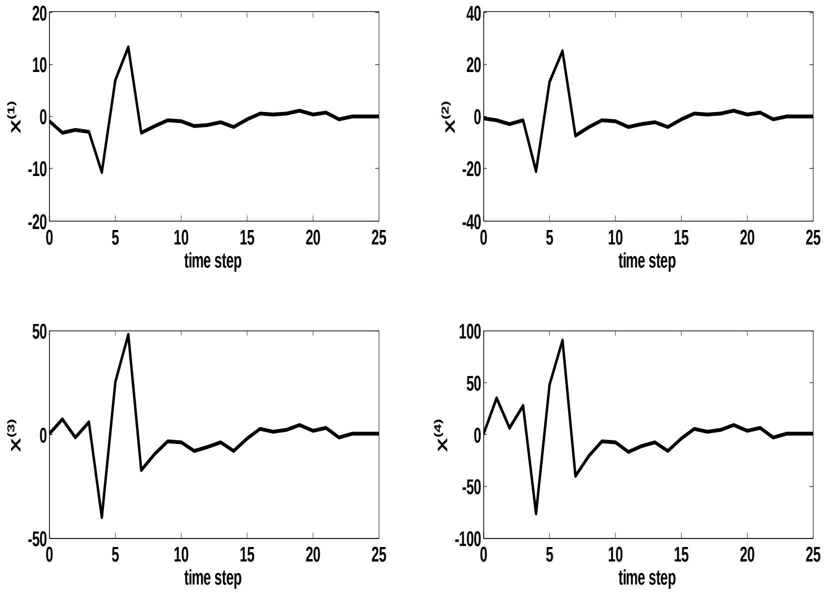

Fig. 2. The state trajectories of the system (23) for the rates $R_1 = R_2 = 3$ bits and $R_3 = R_4 = 2$ bits.

IV. SIMULATION RESULTS

For the purpose of illustration, in this section we consider the closed loop feedback system of Fig. 1 described by the packet erasure channel with erasure probability $\alpha = 0.45$ and the following dynamic system

$$X_{t+1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -16 & 27.3136 & -19.3136 & 6.8284 \end{pmatrix} X_t + U_t, \quad X_0^{(j)} \in [-1, 1], \quad j = 1, 2, 3, 4,$$

$$Y_t = X_t, \quad U_t = \begin{pmatrix} U_t^{(1)} \\ U_t^{(2)} \\ U_t^{(3)} \\ U_t^{(4)} \end{pmatrix}, \quad X_t = \begin{pmatrix} X_t^{(1)} \\ X_t^{(2)} \\ X_t^{(3)} \\ X_t^{(4)} \end{pmatrix}. \tag{23}$$

In the dynamic system (23) the system matrix has four eigenvalues: $\sqrt{2} \pm \sqrt{-1}\sqrt{2}, 2, 2$; and hence, this system without implementing a stabilizing controller is unstable. Note that in this system, the initial state $X_0 = (\, X_0^{(1)} \quad X_0^{(2)} \quad X_0^{(3)} \quad X_0^{(4)} \,)'$ is unknown for the decoder and controller.

To stabilize this system, we use the proposed encoder, decoder and controller of the previous section with rates $R_1 = R_2 = 3$ bits and $R_3 = R_4 = 2$ bits; and we notice that for the system (23) we have

$$\Phi = \begin{pmatrix} -16.3622 & 27.6313 & -15.441 & 2.8619 \\ -16.0947 & 16.176 & -4.1529 & 0.0457 \\ 315.1528 & -388.468 & 195.8491 & -41.3902 \\ 32.4076 & -39.1373 & 19.5672 & -4.0539 \end{pmatrix},$$

$$H_0^{(1)} = 60, \quad H_0^{(2)} = 36, \quad H_0^{(3)} = 900, \quad H_0^{(4)} = 91.$$

Note that the upper bounds $H_0^{(j)}$s on $|Z_0^{(j)}|$ are obtained by varying $X_0^{(j)}$, $j = 1, 2, 3, 4$, in the interval $[-1, 1]$ and computing $Z_0^{(j)}$ using the following equation $Z_0^{(j)} = \Phi X_0$. Fig. 2 illustrates the state trajectories of the system (23) and Fig. 3 illustrates the control trajectories when the proposed encoder, decoder and controller are used with rates $R_1 = R_2 = 3$ bits and $R_3 = R_4 = 2$ bits. As is clear from Fig. 2, using the proposed encoder, decoder and controller, the system (23) is almost sure asymptotically stable. This result is expected as the conditions (13) and (19) hold.

Fig. 4 illustrates the state trajectories of the system (23) when the proposed encoder, decoder and controller with rates $R_1 = R_2 = 2$ bits and $R_3 = R_4 = 2$ bits are used. As is clear from Fig. 4 the proposed encoder, decoder and controller with these rates are not able to stabilize the system. We can expect this result as the condition (19) does not hold for this case.

V. Conclusion

This paper was concerned with the stability of linear time invariant dynamic systems over the packet erasure channel subject to minimum bit rate constraint when encoder and decoder are unaware of control signal. Under this assumption, an encoder, decoder and a controller were co-designed for linear time invariant dynamic systems over the packet erasure channel with feedback acknowledgment that results in almost sure asymptotic stability if a condition relating transmission rate to packet erasure probability and eigenvalues of the system matrix A holds. When the eigenvalues of the system are real valued, it was shown that the obtained sufficient condition is tight. Simulation result illustrated the satisfactory performance of the proposed encoder, decoder and controller for almost sure asymptotic stability.

For linear dynamic systems, the availability of control signal for both encoder and decoder simplifies the design of communication part (encoder and decoder) and controller because under this assumption, the design of communication part and controller can be done separately; and

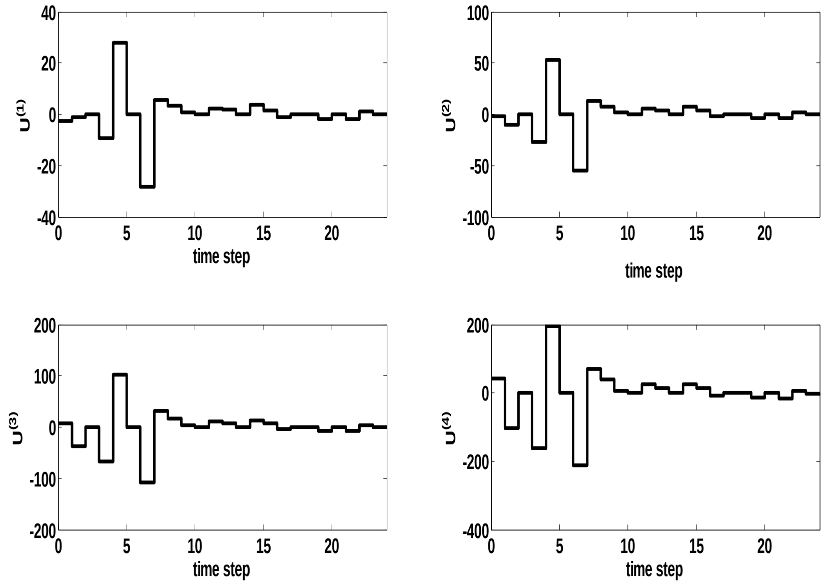

Fig. 3. The control trajectories for the rates $R_1 = R_2 = 3$ bits and $R_3 = R_4 = 2$ bits.

a differential coding technique and a certainty equivalent controller with any stabilizing gain results in stability with minimum required transmission rate [11]-[13]. But, as shown in this paper when encoder and decoder are unaware of control signal, the design of communication and control parts are more complicated because the separation approach [11]-[13] does not work and we have to co-design communication and control parts. Thus, the main difference between the problem solved in this paper and similar problem addressed in [11]-[13] is that encoder and decoder are unaware of control signal and hence the approach used in this paper is based on co-designing communication and control parts; instead of separation design approach used in [11]-[13] which works only if encoder and decoder are aware of control signal.

As discussed, the results of this paper are particularly useful in the development of networked control systems with geographically separated sensors from the controlled system. Examples of such systems are the scenarios discussed in this paper for the coordination of autonomous vehicles. These vehicles have nonlinear smooth dynamics; and therefore, for future research, their dynamics can be linearized around working points and the results of this paper can be applied to address the stability problem associated with the coordination system of autonomous vehicles. For future, it is also interesting to address the stability problem of the block diagram of Fig. 1 described by nonlinear dynamic systems. These problems are left for future investigation.

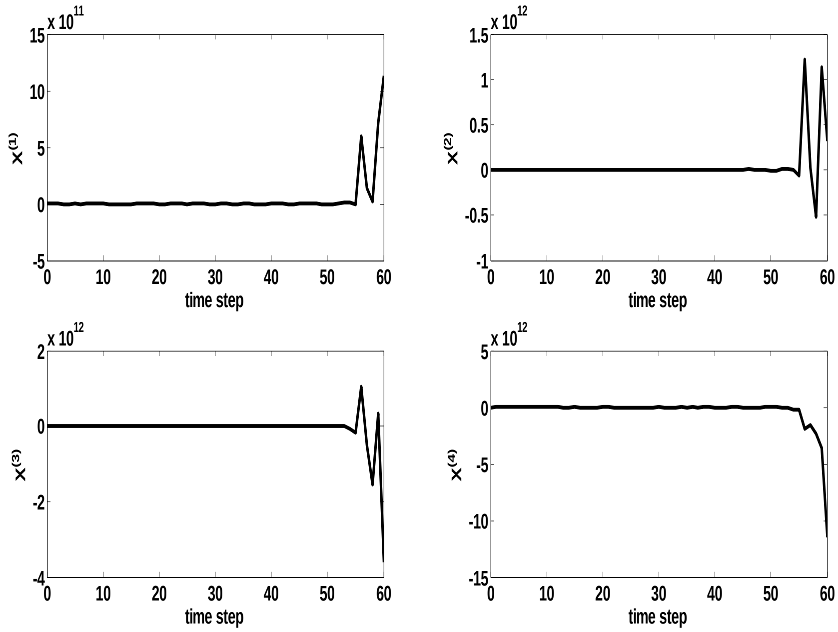

Fig. 4. The state trajectories of the system (23) for the rates $R_1 = R_2 = 2$ bits and $R_3 = R_4 = 2$ bits.

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