Control of Tele-Operation Systems Subject to Capacity Limited Channels and Uncertainty

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Abstract

This paper is concerned with asymptotic observability and stabilizability of dynamical systems over communication channels subject to limited transmission capacity constraint and uncertainty. In particular, the following control/communication systems are considered. 1) The control/communication system described by a controlled diffusion process over an Additive White Gaussian Noise (AWGN) flat fading channel subject to limited transmission capacity constraint, in which, the sensors are geographically separated from the plant while the controller is attached to the plant, and 2) The control/communication system described by an uncertain plant over uncertain communication channels subject to limited transmission capacity constraint, in which, there is communication constraint in both forward and feedback paths.

Keywords — *Networked control systems, stabilizability and observability, robust control.*

1 Introduction

One of the issues that has begun to emerge in some developing application areas such as sensor networks [1] and large scale Networked Control Systems (NCS's) [2] which consists of many components, is how to transmit information and control a plant by communicating information reliably, through communication channels subject to limited transmission capacity constraint. Although the total capacity of these application areas may be large, the resources available for communication between the components can be very limited due to the size or cost. Therefore, in this application areas, it is fundamental to construct encoders, decoders and controllers under limited transmission capacity constraint. One logical approach to address the above question is to find a condition in the form of necessary and sufficient condition in terms of a lower bound for the transmission capacity for reliable communication and/or controlling purposes (i.e., finding the minimum achievable transmission capacity); and then to construct the actual encoder and decoder that can achieve this lower bound. The first step to find the minimum achievable transmission capacity is to derive necessary and sufficient conditions for existence of the encoder and decoder that can work under limited transmission capacity constraint. Some of the fundamental results for stabilizability of NCS's can be found in [3]-[7].
The objective of this paper is twofold. First, to extend the results of [8] and [9] (which is the scalar version of [8]) to the case when the sensors are geographically separated from the controlled system (See Fig. 1). Second, to extend the results presented in [10] to the case when there is uncertain communication channel subject to limited transmission capacity constraint in both forward path (e.g., from the sensor to the controller) and feedback path (e.g., from the controller to the plant) (See Fig. 2).

In [8] and [9] an encoder, decoder and controller are designed for bounded asymptotic and asymptotic observability (e.g., reliable communication) and stabilizability in the mean square sense under limited transmission capacity constraint. The proposed encoder, which encodes the observed information obtained by sensors, is a function of the decoder output while the controller is attached to the plant. Further, in the case of AWGN channel, the proposed encoding/decoding scheme achieves the minimum achievable transmission capacity for bounded asymptotic and asymptotic observability in the mean square sense. Nevertheless, the dependency of the proposed encoder to the decoder output requires the sensors to be also attached to the plant to observe the decoder output by observing the control signal been applied. In contrast, in the control/communication system of Fig. 1, we relax this assumption, to extend the results to a practical case when the sensors are geographically separated from the plant while the controller is attached to the plant. As an example of NCS's which can be described by the control/communication system of Fig. 1, we can recall the tracking problems in which, the trajectory of several moving objects is observed by a geographically separated camera where the observed information is transmitted through a limited transmission capacity wireless communication channel to the local controller attached to each moving objects. [11] introduces a platform which can be described by this tracking problem.

In the control/communication system of Fig. 1, the plant is described by a continuous time controlled diffusion process and the channel is an AWGN flat fading wireless channel. This system can be viewed as a basic block diagram of NCS's described by diffusion processes and AWGN flat fading channels, in which, the sensors are geographically separated from the controlled system (e.g., plant with its attached controller). It can be also viewed as a simplified model of tele-operation system subject to limited transmission capacity constraint in which, there is only communication constraint in forward path. In the spacial case of no control action, the control/communication system of Fig.
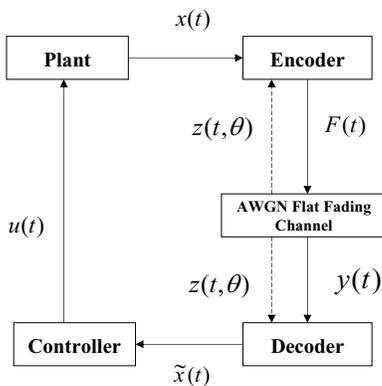
Figure 1. Control/communication system over AWGN flat fading channel

Figure 2: Tele-operation system subject to uncertainty

Figure 3: A basic block diagram of sensor network

1 is reduced to the communication system of Fig. 3 which can be viewed as a basic block diagram of sensor network described by AWGN flat fading channels.

Using a fixed linear encoder, a condition in the form of necessary and sufficient condition for bounded asymptotic and asymptotic observability of the control/communication system of Fig. 1 in the mean square sense is derived. Then, using a certainly equivalent controller, a sufficient condition for bounded asymptotic and asymptotic stabilizability of the control/communication system of Fig. 1 in the mean square sense is given. Further, applying the Bode integral formula, a necessary condition for bounded asymptotic stabilizability in the mean square sense is presented.

On the other hand, in [10], by assuming that there is only an uncertain communication channel subject to limited transmission capacity constraint in forward path, necessary conditions for uniform asymptotic observability and stabilizability in probability and r-mean are derived. In contrast, in the control/communication system of Fig. 2, we are concerned to the case when there is also an uncertain communication channel subject to limited transmission capacity constraint in feedback path. The control/communcation system of Fig. 2 can be viewed as a basic block diagram of NCS's in which, the plant is controlled by a remote controller. It can be also viewed as a basic block diagram of tele-operation system subject to limited transmission capacity constraint and uncertainty. In the control/communication system of Fig. 2 by applying a robust version of the converse of the information transmission theorem and a robust version of the generalized Shannon lower bound, necessary conditions for uniform asymptotic stabilizability in probability and r-mean are presented.

This paper is organized as follows. Section 2 is concerned with the control/communication system of Fig. 1. Necessary and sufficient conditions for bounded asymptotic and asymptotic observability and stabilizability in the mean square sense are presented. In Section 3 the con-

trol/communication system of Fig. 2 is considered and necessary conditions for uniform asymptotic stabilizability in probability and r-mean are given.
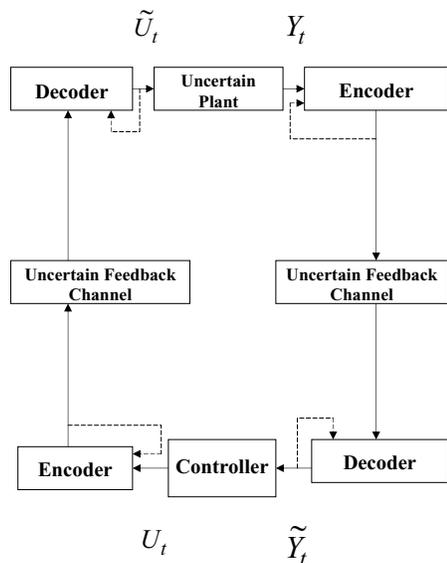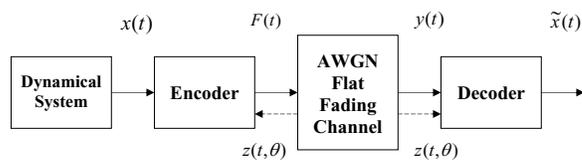
2 Control of Continuous-Time Linear Gaussian Systems over Additive Gaussian Wireless Flat Fading Channels

Consider the control/communication system of Fig. 1. Here, it is assumed that at time t the output of the plant, $x(t) \in \Re^n$, is the information obtained by sensors and it is square integrable, $F(t) \in \Re$ is the encoder output which is also square integrable, $y(t) \in \Re$ is the channel output, $\tilde{x}(t) \in \Re^n$ is the reproduced information at the decoder end and $u(t) \in \Re^m$ is the control signal. Throughout this section, it is assumed that the encoder and decoder know the channel state information, $z(t, \theta)$, which is a function of the random process θ, the encoder at time t is a non-anticipative function of the sample paths x and θ, the decoder at time t is a non-anticipative function of the sample paths y and θ, and the controller at time t is a non-anticipative function of the sample paths $\tilde{x}$ and θ.

The state of the plant is described by the following con-

tinuous time controlled diffusion process given by the Itô equation

$$dx(t) = Ax(t)dt + B(t)u(t)dt + G(t)dw(t), \quad x(0), \quad (1)$$

where $A \in \Re^{n \times n}$ has distinct eigenvalues, $B : [0,T] \to \Re^{n \times m}$, and $G : [0,T] \to \Re^{n \times l}$ are Borel measurable and bounded, and $x(0)$ is Gaussian random variable $x(0) \sim N(\bar{x}_0, \bar{V}_0)$, which is independent of the Gaussian standard Brownian motion w. Throughout this section, it is assumed that the encoder is subject to the instantaneous power constraint

$$E\left[|F(t,x,\theta)|^2 \middle| \theta\right] \leq P \quad (2)$$

and the channel is an AWGN flat fading wireless channel described by the following stochastic differential equation

$$dy(t) = z(t,\theta)F(t,x,\theta)dt + dv(t), \quad y(0) = 0, \quad (3)$$

where v is the Gaussian standard Brownian motion independent of w and $x(0)$. In ([9], Lemma 4.3), it is shown that the transmission capacity of the AWGN flat fading channel (3) in nats per second is

$$C_a = \lim_{T \to \infty} \frac{P}{2T} \int_0^T E_\theta[z^2(t,\theta)]dt, \quad (4)$$

where $E_\theta[.]$ denotes the expectation with respect to the sample path θ.

The objective of this section is to find a condition in the form of a necessary and sufficient condition for bounded asymptotic and asymptotic observability and stabilizability of the control/communication system of Fig. 1 in the mean square sense, defined as follows.

Definition 2.1: (Bounded Asymptotic and Asymptotic Observability in the Mean Square Sense). Define $V(t,y,\theta) \stackrel{\triangle}{=} E\left[(x(t) - \tilde{x}(t,y,\theta))'(x(t) - \tilde{x}(t,y,\theta))\middle|\mathcal{F}_{0,t}^{y,\theta}\right]$, where $\mathcal{F}_{0,t}^{y,\theta} = \mathcal{F}_{0,t}^y \bigvee \mathcal{F}_{0,t}^\theta$, in which, $\mathcal{F}_{0,t}^y$ and $\mathcal{F}_{0,t}^\theta$ are complete filtration generated by $\mathcal{F}_{0,t}^y \stackrel{\triangle}{=} \sigma\{y(s); 0 \leq s \leq t\}$, and $\mathcal{F}_{0,t}^\theta \stackrel{\triangle}{=} \sigma\{\theta(s); 0 \leq s \leq t\}$, respectively ($\sigma$ denotes the sigma algebra). Then, the control/communication system of Fig. 1 is bounded asymptotic (resp. asymptotic) observable in the mean square sense, if for a given control sample path u, there exists an encoder and decoder such that $\lim_{t \to \infty} V(t,y,\theta) < \infty$ P-a.s. (resp. $\lim_{t \to \infty} V(t,y,\theta) = 0$, P-a.s.)

Definition 2.2: (Bounded Asymptotic and Asymptotic Stabilizability in the Mean Square Sense). The control/communication system of Fig. 1 is bounded asymptotic (resp. asymptotic) stabilizable in the mean square sense, if there exists an encoder, decoder, and a controller such that $\lim_{t \to \infty} E\left[||x(t)||_Q^2 \middle| \mathcal{F}_{0,t}^\theta\right] < \infty$, P-a.s., (resp. $\lim_{t \to \infty} E\left[||x(t)||_Q^2 \middle| \mathcal{F}_{0,t}^\theta\right] = 0$, P-a.s.), where $Q = Q' > 0$, and $||x(t)||_Q^2 \stackrel{\triangle}{=} x'(t)Qx(t)$.

2.1 Necessary and sufficient Conditions for Observability

In this section, unlike [8] and [9], we fix the encoder and we find a condition in the form of necessary and sufficient condition for bounded asymptotic and asymptotic observability of the control/communication system of Fig. 1 in the mean square sense.

Since applying a similarity transformation does not change the observability and stabilizability features of the control/communication system of Fig. 1, we apply the similarity transformation $\gamma(t) \stackrel{\triangle}{=} Sx(t)$ on system (1), in which, $SAS^{-1} = \Lambda = diag(\lambda_1(A), ..., \lambda_n(A))$ and $\lambda_i(A)$ are the distinct eigenvalues of A, to obtain the following transformed system

$$d\gamma(t) = \Lambda\gamma(t)dt + SB(t)u(t)dt + SG(t)dw(t),$$
$$\gamma(0) = S\bar{x}_0. \quad (5)$$

Throughout of this section, we assume $SG(t)G'(t)S'$ and $S\bar{V}_0S'$ are diagonal matrices. Under these assumptions, it can be shown that the mean square estimation error, $V(t,y,\theta)$, obtained by transmitting $\gamma(t)$ is diagonal.

Remark 2.3: Notice that if $G(t)$ is orthogonal (e.g, $G(t)G'(t) = I$), A is positive semi-definite, and $\bar{V}_0 = \alpha I$, $\alpha \geq 0$, then $SG(t)G'(t)S'$ and $S\bar{V}_0S'$ are diagonal.

The encoder and the optimal decoder that minimizes the mean square estimation error, $V(t,y,\theta)$, are given in the following theorem.

Theorem 2.4: Consider the control/communication system of Fig. 1 described by (5). Then, the encoder, the optimal decoder, and the corresponding optimal mean square estimation error, are given by

$$F(t,\gamma,\theta) = \sum_{i=1}^n f_{ii}^*(t,\tilde{\gamma}^*,\theta)\gamma_i(t),$$

$$f_{ii}^*(t,\tilde{\gamma}^*,\theta) = \sqrt{\frac{\alpha_i P}{V_{ii}^*(t,y,\theta)}}, \quad (6)$$

$$d\tilde{\gamma}^*(t,y,\theta) = \Lambda\tilde{\gamma}^*(t,y,\theta)dt + SB(t)u(t)dt$$
$$+ z(t,\theta)[\sqrt{\alpha_1 PV_{11}^*}, ..., \sqrt{\alpha_n PV_{nn}^*}]'dy(t)$$
$$- z(t,\theta)\sum_{i=1}^n \sqrt{\frac{\alpha_i P}{V_{ii}^*}}\tilde{\gamma}_i^*(t,y,\theta) \quad (7)$$

$$V_{ii}^*(t,y,\theta) = [S\bar{V}_0S']_{ii} \exp\{2\int_0^t \lambda_i(A)ds$$
$$- \int_0^t \alpha_i z^2(s,\theta)Pds\} + \int_0^t [S(s)G(s)G'(s)S'(s)]_{ii}$$
$$\cdot \exp\{2\int_s^t \lambda_i(A)du - \int_s^t \alpha_i z^2(u,\theta)Pdu\}ds, \quad (8)$$

where $\tilde{\gamma}_i^*(t,y,\theta)$ is the i-th element of $\tilde{\gamma}^*(t,y,\theta) \in \Re^n$, $[W]_{ii} \in \Re^{n \times n}$ denotes the i-th diagonal element of the square matrix W, and the constants $0 \leq \alpha_i \leq 1$, $\sum_{i=1}^n \alpha_i = 1$ are chosen such that $V_{ii}^*(t,y,\theta)$ is bounded asymptotically

(in the case of $G(t) \neq 0$) or asymptotically zero (in the case of $G(t) = 0$).

Proof: After fixing the encoder in the form of the encoder given in (6), following the same methodology used in ([12], Section 16.4), the results are obtained.

Next, from (8), we have the following necessary and sufficient conditions for bounded asymptotic and asymptotic observability in the mean square sense.

Theorem 2.5: i) When the encoder is described by (6) and $G(t) \neq 0$, a necessary and sufficient condition for bounded asymptotic observability in the mean square sense is P-a.s.

$$\frac{P}{2}z^2(t, \theta) \geq \sum_{\{i; Re(\lambda_i(A)) \geq 0\}} Re(\lambda_i(A)), \ a.e. - t \geq 0, \quad (9)$$

ii) When the encoder is described by (6) and $G(t) = 0$, (9) is a necessary and sufficient condition for asymptotic observability in the mean square sense.

Corollary 2.6: i) For the case of AWGN channel (e.g., $z(t, \theta) = 1$) for which the channel capacity is $C_a = \frac{P}{2}$, a necessary and sufficient condition for bounded asymptotic and asymptotic observability in the mean square sense is given by

$$C_a \geq \sum_{\{i; Re(\lambda_i(A)) \geq 0\}} Re(\lambda_i(A)), \ P - a.s. \quad (10)$$

That is, the lower bound (10) is the minimum achievable transmission capacity for bounded asymptotic and asymptotic observability in the mean square sense, in which this bound is achieved by the encoding/decoding scheme proposed in Theorem 2.4.

ii) For the communication system of Fig. 3 described by the encoder (6), a necessary and sufficient condition for bounded asymptotic and asymptotic reliable communication in the mean square sense, as defined by Definition 2.1, is given by (9).

iii) For the communication system of Fig. 3 described by the encoder (6) and AWGN channel (e.g., $z(t, \theta) = 1$), the minimum achievable transmission capacity for bounded asymptotic and asymptotic reliable communication in the mean square sense, is given by (10) in which this capacity is achieved by the encoding/decoding scheme proposed in Theorem 2.4, with $u(t) = 0$.

2.2 Necessary and Sufficient Conditions for Stabilizability

In this section, we propose an output feedback controller for the time-invariant analogous of system (5), in which, it minimizes the following quadratic pay-off for a fixed sample path θ.

$$\bar{J} = \lim_{T \to \infty} \frac{1}{T} E\left\{ \int_0^T [\gamma^{'}(t)Q\gamma(t) + u^{'}(t)Ru(t)]dt \right\}, \quad (11)$$

where $Q = Q^{'} > 0$ and $R = R^{'} > 0$. Here, it is assumed that the system (5) is exponentially stabilizable.

According to the classical separation theorem of estimation and control, the optimal controller that minimizes (5) subject to the AWGN flat fading channel and linear encoder (6) is separated into a state estimator and the certainly equivalent controller given by

$$u^*(t) = -\bar{K}\tilde{\gamma}^*(t, y, \theta), \ \bar{K} = R^{-1}B^{-1}\bar{P},$$
$$0 = Q - \bar{P}SBR^{-1}B^{'}S^{'}\bar{P} + 2\Lambda\bar{P}, \quad (12)$$

where $\tilde{\gamma}^*(t, y, \theta)$ is given by (7). Further, the average criteria is given by

$$\bar{J} = trac\left[\bar{P}SGG^{'}S^{'} + \bar{V}^*\bar{K}^{'}R\bar{K} \right]$$
$$\bar{V}^* = \lim_{t \to \infty} diag\{V_{11}^*(t, y, \theta), ..., V_{nn}^*(t, y, \theta)\}. \quad (13)$$

Subsequently, we have the following theorem for bounded asymptotic and asymptotic stabilizability in the mean square sense.

Theorem 2.7: Consider the time-invariant analogous of system (5) and assume it is exponentially stabilizable. Then, for a fixed sample path θ we have the followings.

i) In the case of $G \neq 0$, using the certainly equivalent controller (12) and the encoding/decoding scheme proposed in Theorem 2.4, the control/communication system of Fig. 1 is bounded asymptotic stabilizable in the mean square sense if P-a.s.

$$\frac{P}{2}z^2(t, \theta) > \sum_{\{i; Re(\lambda_i(A)) \geq 0\}} Re(\lambda_i(A)), \ a.e - t \geq 0. \quad (14)$$

ii) In the case of $G = 0$, using the certainly equivalent controller (12) and the encoding/decoding scheme proposed in Theorem 2.4, the control/communication system of Fig. 1 is asymptotic stabilizable in the mean square sense if (14) holds.

Proof: From (13), it follows that under assumption (14) by using the optimal controller (12) and the encoding/decoding scheme proposed in Theorem 2.4, $\bar{J}$ is bounded (in the case of $G \neq 0$) or zero (in the case of $G = 0$). Subsequently, from (11), it follows that $E||\gamma(t)||_Q^2$ and $E||u(t)||_R^2$ must be asymptotically bounded (in the case of $G \neq 0$) or asymptotically zero (in the case of $G = 0$).

Corollary 2.8: For the case of AWGN channel, a sufficient condition for bounded asymptotic and asymptotic stabilizability in the mean square sense is given by

$$C_a > \sum_{\{i; Re(\lambda_i(A)) \geq 0\}} Re(\lambda_i(A)), \ P - a.s. \quad (15)$$

Remark 2.9: For the case when the encoder, decoder, and controller are linear time-invariant and $z(t, \theta) = 1$, from the Bode integral formula follows that

$$C_a \geq \sum_{\{i; Re(\lambda_i(A)) \geq 0\}} Re(\lambda_i(A)) \quad (16)$$

is a necessary condition for bounded asymptotic stabilizability in the mean square sense.

3 Robust Control over Uncertain Communication Channels

Consider the control/communication system of Fig. 2. Unlike Section 2, the plant, communication channels, controller, and subsequently, the encoders and decoders are discrete in time. Denote by $\mathcal{D}_t$ the set of joint density functions corresponding to a sequence of random variables with length t. Consider the following nominal state space form

$$(\Omega, \mathcal{F}(\Omega), \{\mathcal{F}\}_{t\geq 0}, P):$$
$$\begin{cases} X_{t+1} &= AX_t + BW_t + NU_t, \quad X_0 = X, \\ Y_t &= CX_t + DV_t + MU_t. \end{cases} \tag{17}$$

where $t \in \mathbf{N}_+ \overset{\triangle}{=} \{0,1,2,...\}$, $X_t \in \Re^n$ is the unobserved (state) process, $Y_t \in \Re^d$ is the observed process, $U_t \in \Re^o$ is the control, $W_t \in \Re^m$, $V_t \in \Re^l$, in which $\{W_t; t \in \mathbf{N}_+\}$ is Independent Identically Distributed (i.i.d.) $\sim N(0, I_{m\times m})$ $\{V_t; t \in \mathbf{N}_+\}$, is i.i.d. $\sim N(0, I_{l\times l})$, $X_0 \sim N(\bar{x}_0, \bar{V}_0)$, $\{W_t, V_t, X_0; t \in \mathbf{N}_+\}$ are mutually independent and $D \neq 0$. (C, A) is detectable and $(A, (BB')^{\frac{1}{2}})$ is stabilizable.

Denote by $g \in \mathcal{D}_t$ and $f \in \mathcal{D}_t$ the density functions corresponding to $Y_{0,t-1} \overset{\triangle}{=} \{Y_k\}_{k=0}^{t-1}$ obtained by (17) and the uncertain plant, respectively. Then, the uncertain plant is described by the following relative entropy uncertainty set

$$\mathcal{D}_{SU}(g) \overset{\triangle}{=} \{f \in \mathcal{D}_t; D(f||g) \leq tR_c, g \in \mathcal{D}_t\}, \tag{18}$$

where $D(.||.)$ is the relative entropy [13] and $R_c \in [0, \infty)$. In the control/communication system of Fig. 2, the communication channels are discrete time. In the case of digital noiseless channel with rate $\mathcal{R}$, delayed noiseless digital channel with rate $\mathcal{R}$, and binary erasure channel with rate $\mathcal{R}$ and the packet erasure probability α which deliver $\mathcal{R}$ bits in each time step with probability 1, 1, and $1 - \alpha$, respectively, the transmission capacity is $\mathcal{C} = \mathcal{R}$, $\mathcal{C} = \mathcal{R}$ and $\mathcal{C} = (1 - \alpha)\mathcal{R}$ bits per time step, respectively [3]. Further, if the channel is binary erasure channel with rate $\mathcal{R}$ and $\alpha \in L \subseteq [0,1]$, in which α is deterministic but unknown, the transmission capacity is $\mathcal{C} = (1 - \alpha_{max})\mathcal{R}$ bits per time step, where α_{max} is $\alpha \in L$ which minimizes $(1 - \alpha)$ over the set L.

Next, consider the following asymptotic stabilizability criteria.

Definition 3.1: The control/communication system of Fig. 2 is uniform asymptotic stabilizable in probability and/or r-mean if there exists encoders, decoders, and controller such that

$$\lim_{t\to\infty} \sup_{f\in\mathcal{D}_{SU}(g)} \frac{1}{t}\sum_{k=0}^{t-1} E\rho(X_k, 0) \leq D_v, \tag{19}$$

where for uniform asymptotic stabilizability in probability $D_v \geq 0$ is arbitrary small and

$$\rho(X_k, 0) \overset{\triangle}{=} \begin{cases} 1 & \text{if } ||X_k - 0||_{C'C} > \delta, \\ 0 & \text{if } ||X_k - 0||_{C'C} \leq \delta, \end{cases}$$

$$\delta \geq 0 \text{ is fixed.} \tag{20}$$

While, for uniform asymptotic stabilizability in r-mean, $r > 0$, $D_v \geq 0$ is fixed and $\rho(X_k, 0) = ||X_k - 0||_{C'C}^r$, $r > 0$.

Next, from a robust version of the converse of the information transmission theorem ([10], Theorem 2.5) and a robust version of the generalized Shannon lower bound ([10], Lemma 2.4), we have the following necessary conditions for uniform asymptotic stabilizability in probability and r-mean.

Theorem 3.2: Consider the control/communication system of Fig. 2 under conditional independence assumption, that is, the blocks of Fig. 2 forms a Markov chain. Denote by $\mathcal{C}_r$ and $\mathcal{C}_f$ the transmission capacity of forward channel and feedback channel, respectively.

Then, i) A necessary condition for uniform asymptotic stabilizability in probability in bits per time step is

$$\mathcal{C}_r \geq \mathcal{H}_{robust}(\mathcal{Y}) - \frac{1}{2}\log[(2\pi e)^d \det \Gamma_g],$$

and

$$\mathcal{C}_f \geq \mathcal{H}(\mathcal{U}) - \frac{1}{2}\log[(2\pi e)^d \det \Gamma_g], \tag{22}$$

where $\mathcal{H}_{robust}(\mathcal{Y})$ is the robust Shannon entropy rate (see [10], Definition 2.1) of the observed process when $U_t = 0$, $\mathcal{H}(\mathcal{U})$ is the Shannon entropy rate of the control process, log stands for logarithm with respect to base 2 and Γ_g is the covariance matrix of the Gaussian distribution $h^*(\xi) \sim N(0, \Gamma_g)$, $(\xi \in \Re^d)$ which satisfies $\int_{||\xi||>\delta} h^*(\xi)d\xi = D_v$ (An expression for $\mathcal{H}_{robust}(\mathcal{Y})$ is given in [10], Proposition 3.7; and $\mathcal{H}(\mathcal{U})$ depends to the type of decoder used).

ii) A necessary condition for uniform asymptotic stabilizability in r-mean in bits per time step is given by

$$\mathcal{C}_r \geq \mathcal{H}_{robust}(\mathcal{Y}) + \log\left(\frac{r}{dV_d\Gamma(\frac{d}{r})e^{\frac{d}{r}}}\left(\frac{d}{rD_v}\right)^{\frac{d}{r}}\right),$$

and

$$\mathcal{C}_f \geq \mathcal{H}(\mathcal{U}) + \log\left(\frac{r}{dV_d\Gamma(\frac{d}{r})e^{\frac{d}{r}}}\left(\frac{d}{rD_v}\right)^{\frac{d}{r}}\right), \tag{24}$$

where $\Gamma(.)$ is the gamma function and V_d is the volume of the unit sphere (e.g., $V_d = Vol(S_d); S_d \overset{\triangle}{=} \{\xi \in \Re^d; ||\xi|| \leq 1\}$).

Proof: Consider the decoder in the forward path and the controller as one block. Also consider the encoder in the feedback path and the plant as another block. Then, under the assumption of the existence of encoders, decoders, and controller that provides uniform asymptotic stabilizability in probability or r-mean, it follows that a robust rate distortion (see [10], Definition 2.3) between the outputs of the plant and the outputs of the controller is obtained. Subsequently, from the robust converse of information transmission theorem ([10], Theorem 2.5) and the robust generalized Shannon lower bound ([10], Lemma 2.4), (21) or (23) is obtained. Next, since the robust Shannon entropy rate

of the controlled system is lower bounded by the robust Shannon entropy rate of open loop system (e.g., $U_t = 0$), in (21) and (23), the robust Shannon entropy rate of the observed process when $U_t = 0$ can be used. On the other hand, since a rate distortion between the plant and the controller is obtained, a rate distortion between the outputs of the controller and the outputs of the plant is also occurred. Subsequently, from the robust converse of information transmission theorem and the robust generalized Shannon lower bound, (22) or (24) is obtained.

Acknowledgment

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