Robust Entropy Rate for Uncertain Sources and its Applications in Controlling Systems Subject to Capacity Constraints*

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Abstract

The robust entropy rate is defined as the maximum of the Shannon entropy rate, when the joint probability density function of the source is unknown. The uncertainty of the source probability density is described via a relative entropy constraint set between the uncertain source probability density and the nominal source probability density. For this class of problems, the explicit solution for the robust entropy rate is presented. Further, the results are applied to specific uncertain sources. For the fully observed uncertain Gauss Markov source, a lower bound is found for the robust entropy rate in terms of the solution of an algebraic Riccati equation of the type arising in the H^{∞} estimation and control. Finally, an application of the robust entropy rate for analyzing uniform asymptotic observability and stabilizability of a control/communication system is given. It is shown that for uniform asymptotic observability and stabilizability of an uncertain controlled system over an uncertain communication link, the required robust information channel capacity must be bounded below by the robust entropy rate.

1 Introduction

The objective of this paper is to extend the notion of entropy and subsequently entropy rate to the case when there is uncertainty in the source distribution. Furthermore, to present an application of this notion in analyzing a control/communication system subject to uncertainty.

The robust entropy is defined as the maximum of the Shannon entropy over a family of sources belonging to an uncertainty set. The explicit solution to the robust entropy is given when the uncertainty set is described by a constraint on the relative entropy between the set of uncertain source Probability Density Functions (PDF's) and the corresponding nominal source PDF. Then, specific examples are worked out to illustrate the theory. Subsequently, an application of the robust entropy rate is presented to address necessary conditions for uniform asymptotic observability and stabilizability of a

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control/communication system subject to uncertainty.
This paper is organized as follows. In Section 2, the robust entropy and the robust entropy rate are defined and the explicit solution to the robust entropy is presented. Furthermore, a lower bound is found for the robust entropy rate of the fully observed uncertain Gauss Markov source. Finally, in Section 3, the connection between the robust entropy rate and uniform asymptotic observability and stabilizability of a control/communication system subject to uncertainty is presented. Previous work is found in [1]-[11].

2 Robust Entropy

Information sources and communication channels, in communication applications are often described via probabilistic models. In the analysis and design of telecommunication systems, entropy and channel capacity are the most important concepts when addressing reliability and performance of communication systems. Since the general goal in this paper is to design systems which perform well under uncertainty in the system dynamic and the channel, our first objective is to extend the information theoretic notion of entropy to its robust analogue. In this Section, we introduce the notion of robust entropy and subsequently robust entropy rate, when the information source, while uncertain, it belongs to a family of probabilistic model. Frequency domain uncertainty is described in [9]. The robust entropy is defined as the maximum of the Shannon entropy over the family of the sources belonging to the uncertainty set. When the uncertainty in source is described by the relative entropy uncertainty set, an explicit solution to the robust entropy is derived. Then, specific examples are presented to illustrate the theory. The results of this Section is used in Section 3 to address the question of uniform observability and stabilizability of an uncertain plant, described via relative entropy uncertainty set, when it is controlled via an uncertain communication channel.

2.1 Robust Entropy: Definition

Let $\mathcal{D}$ denote the space of PDF's. Assume the source induces a PDF which belongs to the uncertainty set $\mathcal{D}_{SU} \subset \mathcal{D}$. Then, we have the following definition for robust entropy and robust entropy rate.

Definition 2.1 *(Robust Entropy and Robust Entropy Rate)*
i) Robust Entropy. Let Y be a R.V. corresponding to the uncertain source, and $f_Y(y) \in \mathcal{D}_{SU} \subset \mathcal{D}$ be the corresponding PDF. The robust entropy associated with the family $\mathcal{D}_{SU}$ is defined by

$$H_{robust}(f_Y^*) \stackrel{\triangle}{=} \sup_{f_Y \in \mathcal{D}_{SU}} H_S(f_Y). \tag{1}$$

ii) Robust Entropy Rate. Let $Y_{0,T-1} \stackrel{\triangle}{=} \{Y_0, Y_1, ..., Y_{T-1}\}$ be a sequences with length T corresponding to the uncertain source outcomes. The robust entropy rate associated with the family $\mathcal{D}_{SU}$ of the joint PDF of $Y_{0,T-1}$ is defined by

$$\begin{aligned} \mathcal{H}_{robust}(\mathcal{Y}) &\stackrel{\triangle}{=} \lim_{T\to\infty} \frac{1}{T} H_{robust}(f^*_{Y_{0,T-1}}), \\ H_{robust}(f^*_{Y_{0,T-1}}) &= \sup_{f_{Y_{0,T-1}} \in \mathcal{D}_{SU}} H_S(f_{Y_{0,T-1}}), \end{aligned} \tag{2}$$

provided the limit exists.

2.2 Relative Entropy Uncertainty Set

Throughout this Section, we are concerned with an uncertain set defined by

$$\mathcal{D}_{SU}(g_Y) \stackrel{\triangle}{=} \{f_Y \in \mathcal{D}; H(f_Y|g_Y) \leq R_c + E^{f_Y}(L(y))\}, \tag{3}$$

where $H(.|.)$ is the relative entropy between two PDF's, $R_c \in (0,\infty)$ is fixed, $g_Y(y) \in \mathcal{D}$ is the fixed nominal PDF, $E^{f_Y}(.)$ denotes expectation with respect to uncertain PDF f_Y and $L(y) \geq 0$.

Typical perturbations allowed under the above relative entropy constraint are perturbations in the mean of the nominal PDF. Such perturbations in the mean can be generated by uncertain dynamics [7]. One example of such a perturbations is given in Section 2.3.2. Next, in the following theorem, the explicit solution to the robust entropy is given when the uncertainty set is described by (3).

Theorem 2.2 *Suppose for some $s \in (0,\infty)$, $\frac{s}{1+s}L(y) - \frac{1}{1+s}\log g_Y(y)$ is bounded from below, $(e^{L(y)}g_Y(y))^{\frac{s}{1+s}} \in L_1(\ell(\Re^d))$, and $\frac{sL(y)-\log g_Y(y)}{1+s}(e^{L(y)}g_Y(y))^{\frac{s}{1+s}} \in L_1(\ell(\Re^d))$, where $L_1(\ell(\Re^d))$ is the space of integrable functions with respect to the Lebesgue measure $\ell(.)$ defined on $(\Re^d, \mathcal{B}(\Re^d))$. Then*
i)

$$\begin{aligned} H_{robust}(f_Y^{*,s^*}) &= \min_{s>0}\{sR_c + (1+s)\log\int_{\Re^d}(e^{L(y)}g_Y(y))^{\frac{s}{1+s}}dy\} \\ f_Y^{*,s} &= \frac{(e^{L(y)}g_Y(y))^{\frac{s}{1+s}}}{\int_{\Re^d}(e^{L(y)}g_Y(y))^{\frac{s}{1+s}}dy}, \end{aligned} \tag{4}$$

in which $s^* > 0$ is the minimizing $s \in (0,\infty)$.
ii) Furthermore, if for some $s \in (0,\infty)$, $[\log(e^{L(y)}g_Y(y))]^2(e^{L(y)}g_Y(y))^{\frac{s}{1+s}} \in L_1(\ell(\Re^d))$ and $[\log(e^{L(y)}g_Y(y))](e^{L(y)}g_Y(y))^{\frac{s}{1+s}} \in L_1(\ell(\Re^d))$, the minimizing $s^* > 0$ is the unique solution of

$$G(s) \stackrel{\triangle}{=} H(f_Y^{*,s}|g_Y) - E^{f_Y^{*,s}}(L(y))\Big|_{s=s^*} = R_c. \tag{5}$$

Moreover, $G(s)$ is non-increasing function of $s \in (0,\infty)$, that is

$$0 \leq G(s)\Big|_{s=s_2} \leq G(s)\Big|_{s=s_1} \leq G(s)\Big|_{s=s^*} = R_c \quad 0 < s^* \leq s_1 \leq s_2. \tag{6}$$

Corollary 2.3 *(Robust Entropy Rate). Let $Y_{0,T-1} \stackrel{\triangle}{=} \{Y_0, Y_1, ..., Y_{T-1}\}$, $Y_i : (\Omega, \mathcal{F}(\Omega)) \to (\Re^d, \mathcal{B}(\Re^d))$, $i = 1,2,...,T-1$ be a sequence with length T of the source outcomes corresponding to the uncertain joint PDF $f_{Y_{0,T-1}}(y_{0,T-1}) \in \mathcal{D}_{SU}(g_{Y_{0,T-1}})$, and $R_c \to TR_c$. Then the robust entropy is given by*

$$\begin{aligned} \mathcal{H}_{robust}(\mathcal{Y}) &\stackrel{\triangle}{=} \lim_{T\to\infty} \sup_{\{f_{Y_{0,T-1}}\in\mathcal{D}; \frac{1}{T}H(f_{Y_{0,T-1}}|g_{Y_{0,T-1}})\leq R_c+\frac{1}{T}E^{f_{Y_{0,T-1}}}(L(y_{0,T-1}))\}} \frac{1}{T}H_S(f_{Y_{0,T-1}}) \\ &= \lim_{T\to\infty}\min_{s>0}\{sR_c + \frac{1+s}{T}\log\int_{\Re^{Td}}(e^{L(y_{0,T-1})}g_{Y_{0,T-1}}(y_{0,T-1}))^{\frac{s}{1+s}}dy_{0,T-1}\}, \end{aligned} \tag{7}$$

(provided the limit exists), and

$$f^{*,s}_{Y_{0,T-1}}(y_{0,T-1}) = \frac{(e^{L(y_{0,T-1})}g_{Y_{0,T-1}}(y_{0,T-1}))^{\frac{s}{1+s}}}{\int_{\Re^{Td}}(e^{L(y_{0,T-1})}g_{Y_{0,T-1}}(y_{0,T-1}))^{\frac{s}{1+s}}dy_{0,T-1}}, \tag{8}$$

where the minimizing $s^* > 0$ is the unique solution of

$$H(f^{*,s}_{Y_{0,T-1}}|g_{Y_{0,T-1}}) - E^{f^{*,s}_{Y_{0,T-1}}}(L(y_{0,T-1}))\Big|_{s=s^*} = TR_c. \tag{9}$$

Example 2.4 *From Corollary 2.3, it follows that if the nominal source PDF, $g_{Y_{0,T-1}}$, is Td-dimensional Gaussian distributed with mean zero, that is, $g_{Y_{0,T-1}} \sim N(0, \Gamma_{Y_{0,T-1}})$, $\Gamma_{Y_{0,T-1}} \in \Re^{Td\times Td}$, and $L(y_{0,T-1}) = \frac{1}{2}(y^T)^{tr}\bar{M}y^T$, $y^T \triangleq (y_0^{tr}, y_1^{tr}, ..., y_{T-1}^{tr})$, $y_i \in \Re^d$, $\bar{M} = diag(M_0, M_1, ..., M_{T-1})$, $M_i \in \Re^{d\times d}$, $M_i \geq 0$, $M_i = M_i^{tr}$, $i = 0, 1, ..., T-1$ and $\bar{\rho}(\Gamma_{Y_{0,T-1}}\bar{M}) < 1$, where $\bar{\rho}(.)$ denotes the spectral radius. Then*

$$\begin{aligned}\mathcal{H}_{robust}(\mathcal{Y}) &\triangleq \lim_{T\to\infty} \sup_{\{f_{Y_{0,T-1}}\in\mathcal{D};\frac{1}{T}H(f_{Y_{0,T-1}}|g_{Y_{0,T-1}})\leq R_c+\ \frac{1}{2T}E^{f_{Y_{0,T-1}}}(\sum_{i=0}^{T-1} y_i^{tr}M_iy_i)\}} \frac{1}{T}H_S(g_{Y_{0,T-1}}) \\ &= \frac{d}{2}\log\frac{1+s^*}{s^*} - \lim_{T\to\infty}\frac{1}{2T}\log\det\bar{\Gamma}_{Y_{0,T-1}} + \mathcal{H}_S(\mathcal{Y}), \end{aligned} \tag{10}$$

(provided the limit exists) where $\bar{\Gamma}_{Y_{0,T-1}} \triangleq I - \Gamma_{Y_{0,T-1}}\bar{M}$ ($I \in \Re^{Td\times Td}$ is the identity matrix), $\mathcal{H}_S(\mathcal{Y})$ is the Shannon entropy rate of the nominal distribution, and $s^* > 0$ is the unique solution of the following

$$R_c = -\frac{d}{2}\log\frac{1+s^*}{s^*} + \frac{d}{2s^*} + \lim_{T\to\infty}\frac{1}{2T}\log\det\bar{\Gamma}_{Y_{0,T-1}} \tag{11}$$

Remark 2.5 *i) If the observation process $\{Y_t; t\in\mathbf{N}_+\}$, $\mathbf{N}_+ \triangleq \{0,1,2,...\}$ of the nominal source is stationary, the limit in Example 2.4 exists.*
ii) For the special case when $\bar{M} = 0$, Example 2.4 is reduced to the results reported in [9], that is

$$\mathcal{H}_{robust}(\mathcal{Y}) = \frac{d}{2}\log(\frac{1+s^*}{s^*}) + \mathcal{H}_S(\mathcal{Y}), \quad R_c = -\frac{d}{2}\log(\frac{1+s^*}{s^*}) + \frac{d}{2s^*}. \tag{12}$$

2.3 Examples

In this Section, first the robust entropy rate is calculated for an uncertain source, when the corresponding nominal source is given by a Gauss Markov model. Then, using dynamic programming, a lower bound is found for the robust entropy rate of the uncertain source. The results of this Section will be used in Section 3 to derive necessary conditions for uniform observability and stabilizability of an uncertain controlled system, when it is controlled over an uncertain communication channel.

2.3.1 Uncertain Partially Observed Gauss Markov Source

Let $f_{Y_{0,T-1}}(y_{0,T-1}) \in \mathcal{D}$ denote the true joint PDF of $Y_{0,T-1} \triangleq \{Y_0, Y_1, ..., Y_{T-1}\}$, $Y_i : (\Omega, \mathcal{F}(\Omega)) \to (\Re^d, \mathcal{B}(\Re^d))$, $i = 1, 2, ..., T-1$ and $g_{Y_{0,T-1}}(y_{0,T-1}) \in \mathcal{D}$ be the nominal joint PDF of $Y_{0,T-1}$ produced by the following state space model.

$$(\Omega, \mathcal{F}(\Omega), \{\mathcal{F}_t\}_{t\geq 0}, P) : \left\{ \begin{array}{rcl} X_{t+1} &=& AX_t + BW_t, \quad X_0 = X, \\ Y_t &=& CX_t + DV_t. \end{array} \right. \tag{13}$$

Here, $t \in \mathbf{N}_+$, $X_t \in \Re^n$ is the unobserved (state) process, $Y_t \in \Re^d$ is the observed process, $W_t \in \Re^m$, $V_t \in \Re^l$, in which $\{W_t; t\in\mathbf{N}_+\}$ is Independent Identically Distributed (i.i.d.)$\sim$

$N(0, I_{m\times m})$, $\{V_t; t \in \mathbf{N}_+\}$, is i.i.d. $\sim N(0, I_{l\times l})$, $X_0 \sim N(0, \bar{V}_0)$, $\{W_t, V_t, X_0; t \in \mathbf{N}_+\}$ are mutually independent and $D \neq 0$. It is assumed that (C, A) is detectable, $(A, (BB^{tr})^{\frac{1}{2}})$ is stabilizable and the joint PDF $f_{Y_{0,T-1}}(y_{0,T-1}) \in \mathcal{D}$ of the uncertain source belongs to the following relative entropy uncertainty set.

$$\mathcal{D}_{SU}(g_{Y_{0,T-1}}) = \{f_{Y_{0,T-1}} \in \mathcal{D}; \frac{1}{T}H(f_{Y_{0,T-1}}|g_{Y_{0,T-1}}) \leq R_c + \frac{1}{2T}E^{f_{Y_{0,T-1}}}(\sum_{i=0}^{T-1} y_i^{tr} M_i y_i)\}. \quad (14)$$

In order to calculate the robust entropy rate of this uncertain source, we recall the following result of [9].

Lemma 2.6 *(Shannon Entropy Rate)*
i) [9] The Shannon entropy rate of the nominal source (13) is

$$\mathcal{H}_S(\mathcal{Y}) \triangleq \frac{d}{2}\log(2\pi e) + \frac{1}{2}\log\det\Lambda_\infty, \quad (15)$$

where Λ_∞ *is given by*

$$\begin{aligned} \Lambda_\infty &= CV_\infty C^{tr} + DD^{tr}, \\ V_\infty &= AV_\infty A^{tr} - AV_\infty C^{tr}[CV_\infty C^{tr} + DD^{tr}]^{-1}CV_\infty A^{tr} + BB^{tr}. \end{aligned} \quad (16)$$

ii) Consider the scalar version of (13), with $n = 1$ *and* $d = 1$*. Then (16) can be solved explicitly and then substituted into (15) to obtain the following result.*

$$\mathcal{H}_S(\mathcal{Y}) \geq \frac{1}{2}\log(2\pi e D^2) + \max\{0, \log|A|\}. \quad (17)$$

In (17), the equality holds when $B = 0$*.*

Next, using the result of Example 2.4 and Lemma 2.6, the robust entropy rate of the uncertain source is given in the following proposition.

Proposition 2.7 *(Robust Entropy Rate) The robust entropy rate of the uncertain source described in this Section, is given in Example 2.4 (equation (10)), in which* $\mathcal{H}_S(\mathcal{Y})$ *is given in Lemma 2.6.*

Remark 2.8 *From the chain rule of the Shannon entropy, it follows that the robust entropy rate of a controlled uncertain source with a corresponding controlled nominal source model (13), is bounded below by the robust entropy rate given in Proposition 2.7.*

2.3.2 Uncertain Fully Observed Gauss Markov Source

In this Section, we are concerned with the following fully observed uncertain source

$$(\Omega, \mathcal{F}(\Omega), \{\mathcal{F}_t\}_{t\geq 0}, P) : X_{t+1} = AX_t + BW_t + B\Gamma_t,\ X_0 = X,\ \ Y_t = X_t, \quad (18)$$

where $t \in \mathbf{N}_+$, $X_t = Y_t \in \Re^d$, $X_0 \sim N(0, \bar{V}_0)$, $W_t \in \Re^m$ is i.i.d. $\sim N(0, I)$ and Γ_t denotes the uncertain random perturbation. The nominal source corresponding to the uncertain source (18) is the source with $\Gamma_t = 0$. Let $f_{Y_{0,T-1}} \in \mathcal{D}$ denotes the joint PDF of a sequence $Y_{0,T-1} \triangleq \{Y_0, Y_1, ..., Y_{T-1}\}$, $Y_i : (\Omega, \mathcal{F}(\Omega)) \to (\Re^d, \mathcal{B}(\Re^d))$, $i = 1, 2, ..., T-1$, with length T of uncertain source outcomes (18) and $g_{Y_{0,T-1}}(y_{0,T-1}) \in \mathcal{D}$ the joint PDF of a sequence with length T of the nominal source. Here, it is assumed that $lim_{T\to\infty}\frac{1}{T}E^{f_{Y_{0,T-1}}}(\sum_{i=0}^{T-2}||\Gamma_i||^2) < \infty$ and $f_{Y_{0,T-1}} \in \mathcal{D}_{SU}(g_{Y_{0,T-1}})$, in which $\mathcal{D}_{SU}(g_{Y_{0,T-1}})$ is given by (14), with $M_i = M \in \Re^{d\times d}$, $M_i = M_i^{tr}$, $M_i \geq 0$, $i = 0, 1, ..., T-2$, and $M_{T-1} = 0$.

Lemma 2.9 *Define* $\mathcal{D}_{SU}(\Gamma_{0,T-2}) \triangleq \Big\{\{\Gamma_i\}_{i=0}^{T-2}; \frac{1}{2T}E^{f_{Y_{0,T-1}}}(\sum_{i=0}^{T-2}||\Gamma_i||^2) \leq R_c + \frac{1}{2T}E^{f_{Y_{0,T-1}}}$ $(\sum_{i=0}^{T-2} Y_i^{tr} M Y_i)\Big\}$. *The robust entropy rate is given by*

$$\begin{aligned}&\mathcal{H}_{robust}(\mathcal{Y})\\&= \lim_{T\to\infty}\frac{1}{T}\sup_{\{\Gamma_{0,T-2}\in\mathcal{D}_{SU}(\Gamma_{0,T-2})\}}\Big\{-\frac{1}{2T}E^{f_{Y_{0,T-1}}}(\sum_{i=0}^{T-2}||\Gamma_i||^2) - E^{f_{Y_{0,T-1}}}(\log g_{Y_{0,T-1}})\Big\}.\end{aligned} \tag{19}$$

Next, by applying dynamic programming, the solution to $\mathcal{H}_{robust}(\mathcal{Y})$ is given in the following proposition.

Proposition 2.10 *Let* $B^{tr}(BB^{tr})^{-1}B < (1+s)I$, (A,B) *is controllable,* A *is invertible, and* $\Psi(\eta) > 0$ *for some* η, $|\eta| = 1$, *where* $\Psi(z)$ *is the rational matrix function given by* $\Psi(z) = \frac{B^{tr}(BB^{tr})^{-1}B}{2} - \frac{1+s}{2}I + B^{tr}(z^{-1} - A^{tr})^{-1}\frac{s}{2}M(z-A)^{-1}B$, $s > 0$. *Then*

$$\mathcal{H}_{robust}(\mathcal{Y}) = \min_{s>0}\{sR_c + \frac{d}{2}\log(2\pi e) + \frac{1}{2}\log\det BB^{tr} + Trac(BB^{tr}\Sigma)\}, \tag{20}$$

where Σ is the steady state solution of the following Riccati equation which is of the type arising in H^∞ control.

$$\begin{aligned}\Sigma_t &= A^{tr}\Sigma_{t+1}A - A^{tr}\Sigma_{t+1}B[\frac{B^{tr}(BB^{tr})^{-1}B}{2} - \frac{1+s}{2}I + B^{tr}\Sigma_{t+1}B]^{-1}B^{tr}\Sigma_{t+1}A + \tilde{M}_t,\\&\quad t = 0,1,...,T-1, \quad \Sigma_{T-1} = 0,\end{aligned} \tag{21}$$

where $\tilde{M}_0 = \frac{s}{2}M_0 + \frac{1}{2}V_0^{-1}$ and $\tilde{M}_i = \frac{s}{2}M$, $i = 1 \geq 1$.

Remark 2.11 *From the chain rule of the Shannon entropy, it can be shown that the robust entropy rate of a controlled version of uncertain source (18) is bounded below by* $\mathcal{H}_{robust}(\mathcal{Y})$, *given in (20).*

3 Applications in Control/Communication Systems

In this Section, general necessary conditions for uniform asymptotic observability and stabilizability in probability and r-mean are presented for the control/communication system given in Figure 1. Then the obtained results are applied to the controlled version of uncertain plants (sources) described in Sections 2.3.1 and 2.3.2 to present necessary conditions for uniform asymptotic observability and stabilizability of these uncertain plants controlled over uncertain communication channel. Throughout this Section, we assume that the control law at time t, $U_t = \mu(t, \tilde{Y}_0, \ldots, \tilde{Y}_t)$, is a non-anticipative functional of the decoder output up to time t. The encoder law at time t, $Z_t = \mathcal{E}(t, Y_0, Y_1, ..., Y_t, Z_0, Z_1, ..., Z_{t-1})$, is a non-anticipative functional of the source output up to time t, and the previous output of the encoder up to time $t-1$. Finally, the decoder law at time t, $\tilde{Y}_t = \mathcal{A}(t, \tilde{Z}_0, \tilde{Z}_1, ..., \tilde{Z}_t, \tilde{Y}_0, \tilde{Y}_1, ..., \tilde{Y}_{t-1})$, is a non-anticipative functional of the channel output up to time t, and the previous output of the decoder up to time $t-1$.

Next, consider the control/communication system of Figure 1. Let $(\Omega, \mathcal{F}(\Omega), P)$ be a complete probability space and $Y_t \in \Re^d$ be the observed output of the uncertain plant obtained by sensors at time t. The objective is to find a necessary condition for uniform asymptotic observability and stabilizability in probability and r-mean defined as follows.

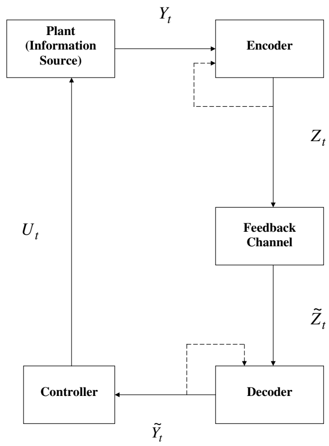

Figure 1: Block diagram of a control/communication system

Definition 3.1 *(Uniform Asymptotic Observability in Probability and r- Mean). Consider the control/communication system of Figure 1. The uncertain plant is uniform asymptotic observable in probability or r-mean over an uncertain communication channel, if there exists an encoder and decoder such that*

$$\lim_{t\to+\infty} \sup_{f_{Y_{0,t-1}}\in\mathcal{D}_{SU}} \frac{1}{t}\sum_{k=0}^{t-1} E\rho(Y_k,\tilde{Y}_k) \leq D_v, \tag{22}$$

where $f_{Y_{0,t-1}}(y_{0,t-1}) \in \mathcal{D}_{SU}$ is the joint PDF of $Y_{0,t-1} \triangleq \{Y_0, Y_1, ..., Y_{t-1}\}$ produced by the uncertain plant; for uniform asymptotic observability in probability, $D_v \geq 0$ is arbitrary small and $\rho(Y_k,\tilde{Y}_k)$ is defined by

$$\rho(Y_k,\tilde{Y}_k) \triangleq \begin{cases} 1 & \text{if } ||Y_k - \tilde{Y}_k|| > \delta, \\ 0 & \text{if } ||Y_k - \tilde{Y}_k|| \leq \delta, \end{cases} \tag{23}$$

where $||.||$ is Euclidian norm and $\delta \geq 0$ is fixed, and for uniform asymptotic observability in r-mean, $D_v \geq 0$ is fixed and $\rho(Y_k,\tilde{Y}_k) = ||Y_k - \tilde{Y}_k||^r, r > 0$.

Next, assume there is a linear relationship between the observed signal, Y_t, and the state variable, X_t, of the uncertain plant. That is, $Y_t = CX_t + \Upsilon_t$, where Υ_t, in general, is subject to uncertainty and it is a function of time, control signal and measurement noises. Under this assumption, the uniform asymptotic stabilizability in probability and r-mean is defined as follow.

Definition 3.2 *(Uniform Asymptotic Stabilizability in Probability and r- Mean). Consider the control/communication system of Figure 1. The uncertain plant is uniform asymptotic stabilizable in probability or r-mean if there exists an encoder, decoder, and controller such that*

$$\lim_{t\to\infty} \sup_{f_{Y_{0,t-1}}\in\mathcal{D}_{SU}} \frac{1}{t}\sum_{k=0}^{t-1} E\rho(X_k,0) \leq D_v, \tag{24}$$

where for uniform asymptotic stabilizability in probability $D_v \geq 0$ is arbitrary small and $\rho(X_k, 0)$ is defined by

$$\rho(X_k, 0) \triangleq \begin{cases} 1 & \text{if } ||X_k - 0||_{C^{tr}C} > \delta, \\ 0 & \text{if } ||X_k - 0||_{C^{tr}C} \leq \delta, \end{cases} \tag{25}$$

where $||x - 0||_{C^{tr}C} \triangleq \left(x^{tr} C^{tr} C x\right)^{\frac{1}{2}}$, and for uniform asymptotic stabilizability in r-mean, $D_v \geq 0$ is fixed and $\rho(X_k, 0) = ||X_k - 0||^r_{C^{tr}C}, r > 0$.

Next, using a robust version of Information Transmission theorem given in [10, 11] and a lower bound for the robust rate distortion (for definition of the robust rate distortion, see [10] or [11]), the main result of this Section is presented in the following theorem.

Theorem 3.3 *i) For uniform asymptotic observability and stabilizability in probability, a necessary condition on the robust channel capacity is*

$$\mathcal{C}_{robust} \quad \geq \quad \mathcal{H}_{robust}(\mathcal{Y}) - \frac{1}{2}\log[(2\pi e)^d \det \Gamma_g], \tag{26}$$

where $\mathcal{H}_{robust}(\mathcal{Y})$ is the robust entropy rate of the observed process (uncertain source) and Γ_g is the covariance matrix of the Gaussian distribution $g^*(y) \sim N(0, \Gamma_g), (y \in \Re^d)$ which satisfies

$$\int_{||y||>\delta} g^*(y)dy = D_v, \tag{27}$$

in which $D_v \geq 0$ is arbitrary small.
ii) A necessary condition for r-mean uniform asymptotic observability and stabilizability is

$$\mathcal{C}_{robust} \geq \mathcal{H}_{robust}(\mathcal{Y}) - \frac{d}{r} + \log\left(\frac{r}{dV_d\Gamma(\frac{d}{r})}\left(\frac{d}{rD_v}\right)^{\frac{d}{r}}\right), \tag{28}$$

where $\Gamma(.)$ is the gamma function and V_d is the volume of the unit sphere (e.g., $V_d = Vol(S_d)$; $S_d \triangleq \{y \in \Re^d; ||y|| \leq 1\}$).

Remark 3.4 *We have the following remarks regarding the above theorem.*
i) The robust entropy rate is a function of the control signal.
ii) For the case $d = 1$, condition (27) is reduced to

$$2\Phi\left(-\frac{\delta}{\sqrt{\Gamma_g}}\right) = D_v, \tag{29}$$

where $\Phi(t) \triangleq \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du$. Using a table for this integral, we notice that for an arbitrary small D_v, $\Gamma_g = \frac{\delta^2}{16}$ should be used in (26).
iii) Finally, it is pointed out that the necessary conditions derived in Theorem 3.3, are practically important because they give flexibility to the designer to relate the observability and stabilizability error to the minimum capacity necessary for observability and stabilizability.
iv) These necessary conditions give tighter bounds than the usual bound that involves only $\mathcal{H}_S(\mathcal{Y})$.

Since lower bounds for (26) and (28) are also necessary conditions for observability and stabilizability, the results of Theorem 3.3 are applicable to the controlled version of the uncertain sources described in Sections 2.3.1 and 2.3.2 in the following corollary.

Corollary 3.5 *i) A necessary condition for uniform asymptotic observability and stabilizability of a controlled version of the uncertain plant (source) described in Section 2.3.1 is given by (26) and (28) respectively, in which $\mathcal{H}_{robust}(\mathcal{Y})$ is given in Proposition 2.7.*
ii) A necessary condition for uniform asymptotic observability of a controlled version of uncertain plant (source) (18) is given by (26) and (28) respectively, in which $\mathcal{H}_{robust}(\mathcal{Y})$ is given in Proposition 2.10.

Next, we have the following corollary as a direct result of Example 2.4 and Theorem 3.3.

Corollary 3.6 *Consider a controlled uncertain partially observed Gauss Markov plant corresponding with the following controlled version of partially observed Gauss Markov system (13), via the relative entropy uncertainty constraint (14),*

$$(\Omega, \mathcal{F}(\Omega), \{\mathcal{F}_t\}_{t\geq 0}, P) : \left\{ \begin{array}{lll} X_{t+1} & = & AX_t + BW_t + NU_t, \quad X_0 = X, \\ Y_t & = & CX_t + DV_t. \end{array} \right. \tag{30}$$

in which $X_t \in \Re^n$, $W_t \in \Re^m$, $U_t \in \Re, Y_t \in \Re$, and $V_t \in \Re^l$. Assume this system is controlled over a linear time-invariant single-input single-output discrete time, additive Gaussian noise stable channel. That is, in compact notation, $\tilde{Y}(z) = H_c(z)Y(z)+W_c(z)$, where $H_c(z)$ (the channel transfer function) has poles inside unit circle and $W_c(z)$ is the frequency response of the channel noise $\{W_c(t); t \in \mathbf{N}_+\}$, which is Additive White Gaussian Noise (AWGN) process with mean zero and variance $\sigma^2_{W_c}$, and it is mutually independent of $\{X_0, W_t, V_t; t \in \mathbf{N}_+\}$. Assume the controller is stable linear time-invariant. That is, the controller transfer function $K_c(z)$ has poles inside unit circle. Moreover, assume the nominal system open loop transfer function $L(z) = P(z)K_c(z)H_c(z)$, $P(z) = C(zI - A)^{-1}N$ is strictly proper transfer function.
An application of Bode integral formula [12] implies that for uniform asymptotic stabilizability in r-mean, the required channel capacity must satisfy

$$\begin{aligned} \mathcal{C}^{cap} &\geq \frac{1}{2}\log(\frac{1+s^*}{s^*}) - \lim_{T\to\infty}\frac{1}{2T}\log\det\bar{\Gamma}_{Y_{0,T-1}} + \sum_{\{i;|\lambda_i(A)|\geq 1\}} \log|\lambda_i(A)| \\ &+\frac{1}{4\pi}\int_{-\pi}^{\pi}\log(F(e^{jw})BB^{tr}F^{tr}(e^{-jw}) + DD^{tr} + |G(e^{jw})|^2\sigma^2_{W_c})dw \\ &+\frac{1}{2}\log(2\pi e) - \Delta, \end{aligned} \tag{31}$$

where $\mathcal{C}^{cap}$ denote the AWGN channel capacity, $s^* > 0$ is given in (11) (with $d = 1$), $\bar{\Gamma}_{Y_{0,T-1}} = I - \Gamma_{Y_{0,T-1}}\bar{M}$, where $\Gamma_{Y_{0,T-1}}$ is the covariance of the sequence Y^T corresponding to the nominal system (30) with $U_t = 0$, $t = 0, 1, 2, ..., T-1$, $F(e^{jw}) = C(e^{jw}I - A)^{-1}$, $G(e^{jw}) = P(e^{jw})K_c(e^{jw})$, $|G(e^{jw})|^2 = G(e^{jw})G^{tr}(e^{jw})$, and $\Delta = \frac{1}{r} - \log\left(\frac{r}{2\Gamma(\frac{1}{r})}(\frac{1}{rD_v})^{\frac{1}{r}}\right)$.

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