Robust Control of a Class of Feedback Systems Subject to Limited Capacity Constraints

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***Abstract*— In this paper we are concerned with uniform mean square reliable data reconstruction and robust stability for a class of dynamical systems over Additive White Gaussian Noise (AWGN) channels, subject to the limited capacity constraints. Specifically, the design of an encoder, decoder and controller subject to the mean square reliable data reconstruction and stability, is considered for a class of dynamical systems. The class of dynamical systems which are described the uncertainty is modeled via a relative entropy constraint.**

I. Introduction

Recently, there has been a significant progress in addressing reliable data reconstruction (known as observability) and stability of dynamical systems which are controlled over limited capacity communication channels [1]-[13] (throughout the capacity is measured in bits per source message which is directly related to the transmission bit rate). In this paper, we are concerned with the control/communication system of Fig. 1. The control/communication system of Fig. 1 is defined on a complete probability space $(\Omega, \mathcal{F}(\Omega), P)$ with filtration $\{\mathcal{F}\}_{t\geq 0}$; $t \in \mathbf{N}_+ \triangleq \{0, 1, 2, ...\}$, where Y_t, Z_t, $\tilde{Z}_t$, $\tilde{Y}_t$ and U_t, $t \in \mathbf{N}_+$ are Random Variables (R.V.'s) denoting the source message, channel input codeword, channel output codeword, the reproduction of the source message, and the control input to the source, respectively. The objective of this paper is to design an encoder, decoder and controller which achieve uniform mean square reconstruction and robust stability for a class of dynamical systems, when the capacity of the communication channel is limited.

The problem of uniform observability and robust stability of fully observed uncertain dynamical systems subject to a bounded disturbance input is considered in [4], [9], [12], [13]. This paper complements the already existing results in the literature since it addresses similar questions for a class of dynamical system, which is described by a relative entropy constraint (the class denotes the uncertainty description of the system). This uncertainty description is a generalization of the sum quadratic uncertainty description considered in [14],[15], [4], [9], [12], [13], and it is shown to have nice structure [16].

This paper is organized as follows. In Section II, the problem formulation and mathematical preliminaries are given. In Section III, an encoding scheme for mean square reliable data reconstruction of an uncertain source which produces orthogonal processes, is proposed. Subsequently, in Section IV, stability for a class of dynamical systems subject to quadratic constraints are investigated.

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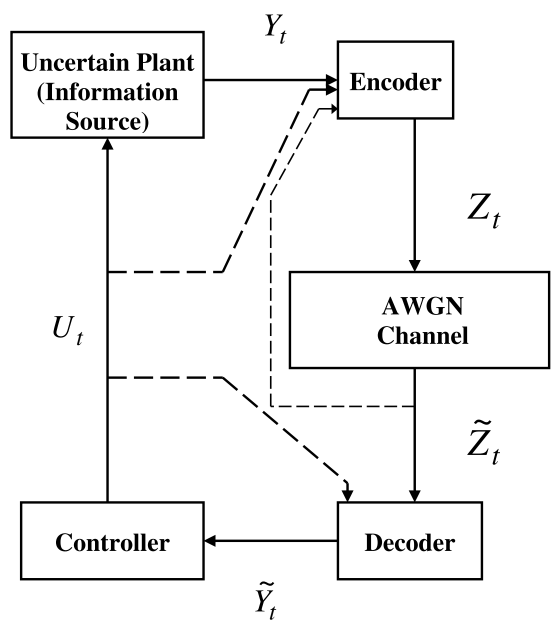

Fig. 1. Control/communication system subject to uncertainty in the source

II. Problem Formulation and Mathematical Preliminaries

In this paper, we are concerned with the control/communication system of Fig. 1. Throughout, sequences of R.V.'s are denoted by $Y^T \triangleq (Y_0, Y_1, ..., Y_T)$ for $T \in \mathbf{N}_+$. $\log(.)$ denotes the natural logarithm and I_q denotes identity matrix with dimension $(q \times q)$. A stochastic kernel $P(dF; x)$ is a mapping $P : \hat{\mathcal{A}} \times A \rightarrow [0, 1]$ which satisfies i) For every $x \in A$, the set function $P(:; x)$ is a probability measure on $\hat{\mathcal{A}}$, and ii) For every $F \in \hat{\mathcal{A}}$, the function $P(dF; .)$ is $\mathcal{A}$-measurable ($(A, \mathcal{A})$, $(\hat{A}, \hat{\mathcal{A}})$ are measurable spaces). $diag(...)$ denotes diagonal matrix, $\sigma\{.\}$ denotes σ-algebra and $\bar{\sigma}(.)$ denotes the biggest singular value.

The different blocks of Fig. 1 are described below.

Information Source. The information source is described by the probability measure $P(dY^T) = f_{Y^T} dY^T$ which depends on the control sequence as shown in Fig. 1. It is assumed that the density function f_{Y^T} belongs to the following relative entropy constraint.

$$f_{Y^{T-1}} \in \mathcal{D}_{SU}(g_{Y^{T-1}}) \stackrel{\triangle}{=} \Big\{ f_{Y^{T-1}}; \frac{1}{T} H(f_{Y^{T-1}} || g_{Y^{T-1}}) \leq R_c + E_{f_{Y^{T-1}}} [\frac{1}{2T} \sum_{t=0}^{T-1} Y_t^{'} M_t Y_t] \Big\} \quad (1)$$

where $H(.||.)$ is the relative entropy [17], $g_{Y^{T-1}}$ and $f_{Y^{T-1}}$ are the joint density functions associated with observation Y^{T-1} obtained from nominal and uncertain systems, respectively, $R_c \geq 0$ and $M_t = M_t^{'} \in \Re^{d \times d}$ is positive semi-definite, and $E_{f_{Y^{T-1}}}[.]$ is the expectation with respect to the joint density function $f_{Y^{T-1}}$.

The relative entropy $H(f_{Y^{T-1}} || g_{Y^{T-1}})$ can be thought of as a measure of the difference between the nominal density function $g_{Y^{T-1}}$ and the perturbed density function $f_{Y^{T-1}}$. Typical perturbation allowed under the above relative entropy constraint are the perturbations in the mean of the density function $g_{Y^{T-1}}$ [18]. One example of such perturbations is given by the following class of fully observed Gauss Markov systems.

$$(\Omega, \mathcal{F}(\Omega), P; \{\mathcal{F}_t\}_{t\geq 0}) : \left\{ \begin{array}{ll} X_{t+1} = AX_t + NU_t + BW_t + B\bar{W}_t, & X_0 = X, \\ Y_t = H_t, & H_t = X_t \end{array} \right. \quad (2)$$

where $X_t \in \Re^d$, $U_t \in \Re^o$, $W_t \in \Re^m$, $\bar{W}_t \in \Re^m$, $X_0 \sim N(\bar{x}_0, \bar{V}_0)$, $H_t \in \Re^d$, W_t is i.i.d. $\sim N(0, \Sigma_W), \Sigma_W > 0$, $\bar{W}_t$ is the perturbed noise random process which is $\{\sigma\{W_l\}; l \leq t-1\}$ adapted, and H_t is the signal to be controlled.

The nominal system associated with the above uncertain system, is the following fully observed system.

$$(\Omega, \mathcal{F}(\Omega), \Pi; \{\mathcal{F}_t\}_{t\geq 0}) : \left\{ \begin{array}{ll} X_{t+1} = AX_t + NU_t + BW_t, & X_0 = X, \\ Y_t = H_t, & H_t = X_t \end{array} \right. \quad (3)$$

It can be shown that for the sequence Y^{T-1}, $H(f_{Y^{T-1}} || g_{Y^{T-1}}) = \frac{1}{2} E_P \Big[\sum_{t=0}^{T-2} \bar{W}_t^{'} \Sigma_W^{-1} \bar{W}_t \Big]$. That is, the relative entropy constraint (1) holds for the uncertain system (2) with the nominal system (3), provided the following sum quadratic constraint holds.

$$E_P \Big[\frac{1}{2T} \Big(\sum_{t=0}^{T-2} (\bar{W}_t^{'} \Sigma_W^{-1} \bar{W}_t) - \sum_{t=0}^{T-1} (X_t^{'} M_t X_t) \Big) - R_c \Big] \leq 0. \quad (4)$$

Remark 2.1: It is clear from above example that the relative entropy uncertainty description (1) gives as a special case a constraint on the energy of the uncertainty. Such uncertainty description has been considered in [14], [15]; and for continuous time systems (in the form of integral quadratic constraint uncertainty description) has been considered in [16], [19]-[21].

Communication Channel: The communication channel is an AWGN channel. That is, the communication channel at time $t \in \mathbf{N}_+$ is described by $\tilde{Z}_t = Z_t + \tilde{W}_t$ ($E[Z_t^{'} Z_t] \leq P_t$), where $Z_t, \tilde{Z}_t \in \Re^d$ are the channel input and output at time t and the stochastic process $\{\tilde{W}_t \in \Re^d; t \in \mathbf{N}_+\} \sim N(0, W_c)$ is an orthogonal zero mean Gaussian process independent of Z_t.

For an AWGN channel $\tilde{Z}_t = Z_t + \tilde{W}_t$, $Z_t \in \Re$, $E(Z_t^2) \leq P_t$, $\tilde{W}_t \sim N(0, W_c)$, the channel capacity with and without feedback is the same, and it is given by $\mathcal{C} = \lim_{T\to\infty} \frac{1}{2T} \sum_{t=0}^{T-1} \log(1 + \frac{P_t}{W_c})$, nats, provided the limit exists.

Encoder: The encoder at any time $t \in \mathbf{N}_+$ is modeled by a stochastic kernel $P(dZ_t; y^t, u^{t-1}, \tilde{z}^{t-1})$.

Decoder: The decoder at any time $t \in \mathbf{N}_+$ is modeled by a stochastic kernel $P(d\tilde{Y}_t; \tilde{z}^t, u^{t-1})$.

Controller: The controller at any time $t \in \mathbf{N}_+$ is modeled by a stochastic kernel $P(dU_t; \tilde{z}^t, u^{t-1})$.

In this paper, we construct encoding and stabilizing schemes which guarantee uniform mean square reconstruction and stability (as defined below) for the class of systems (2) when the perturbed noise process is subject to the sum quadratic constraint (4).

Definition 2.2: (Uniform Mean Square Reconstruction). Consider the control/communication system of Fig. 1 over a class of dynamical systems. The signal Y^T is uniformly reconstructed using a mean square error criterion if there exist a control sequence, an encoder and decoder such that

$$\lim_{T\to\infty} \sup_{f_{Y^{T-1}} \in \mathcal{D}_{SU}(g_{Y^{T-1}})} \frac{1}{T} \sum_{t=0}^{T-1} E||Y_t - \tilde{Y}_t||^r \leq D_v, \quad (5)$$

for $r = 2$ and a finite $D_v \geq 0$.

Definition 2.3: (Mean Square Robust Stability). Consider the control/communication system of Fig. 1 over a class of dynamical systems. Let $Y_t = H_t + \Gamma_t$ where H_t is the signal to be controlled and Γ_t is a function of measurement noise and uncertainty. The signal H^T is mean square stabilizable if there exists an encoder, decoder and controller such that

$$\lim_{T\to\infty} \sup_{f_{Y^{T-1}} \in \mathcal{D}_{SU}(g_{Y^{T-1}})} \frac{1}{T} \sum_{t=0}^{T-1} E||H_t||^r \leq D_v, \quad (6)$$

for $r = 2$ and a finite $D_v \geq 0$.

In [10], a general necessary condition for uniform observability and robust stability of an uncertain system described via the relative entropy constraint (1) was derived. It is based on the suitable application of the following lower bound relating capacity and rate distortion.

Theorem 2.4: Consider a communication system without feedback equipped with an encoder and decoder (similar to Fig. 1 without feedback) in which $Y_t \in \Re^d$. A necessary condition for r-mean uniform reconstruction of Y^T is given by

$$\mathcal{C} \geq \mathcal{H}_r(\mathcal{Y}) - \frac{d}{r} + \log(\frac{r}{dV_d \Gamma(\frac{d}{r})} (\frac{d}{rD_v})^{\frac{d}{r}}) \stackrel{\triangle}{=} R_{S,r}(D_v), \quad (7)$$

where $\mathcal{C}$ is the channel capacity measured in nats per source message, $\mathcal{H}_r(\mathcal{Y})$ is the entropy rate for a class of sources (see [10], Definition 2.1, in which the quantity is defined as a maximization over the class of source of the entropy rate), $\Gamma(.)$ is the gamma function, V_d is the volume of the unit sphere (e.g., $V_d = Vol(S_d)$; $S_d \stackrel{\triangle}{=} \{y \in \Re^d; ||y|| \leq 1\}$) and $R_{S,r}(D_v)$ is the robust Shannon lower bound (see [10], Lemma 2.4).

Although, Theorem 2.4 is subject to the case of without feedback, the results can be extended to feedback channels and sources which use feedback from the output of the decoder to the input of the encoder (hence they are applicable to the system of Fig. 1), provided 1) The capacity of the channel with and without feedback are the same, 2) the rate distortion of a source without using feedback from the decoder to the encoder is the same as the one that uses such feedback, and the reconstruction kernel is causal. Conditions 1) is valid for the AWGN channel considered, while condition 2) holds if the source output is an orthogonal process, which uses feedback.

Note that subject to conditions 1), 2) above, then (7) is also a necessary condition for uniform reconstruction and stability of an innovation type encoder using definitions (5) and (6), when $\lim$ is replaced with $\limsup$.

The results given in subsequent sections complement our previous results reported in [10] by proposing encoding schemes which guarantee uniform mean square observability by transmitting $\mathcal{C} = R_{S,r}(D_v)$ bits per time, over AWGN channels. From Theorem 2.4 it follows that this rate is the minimum possible capacity for uniform mean square reconstruction.

III. Uncertain Source Described via Relative Entropy Constraint

In this section we first compute the robust rate distortion for a class of pre-processes sources whose encoder output is a class of orthogonal processes, in which the class is described via the relative entropy constraint (1), having a Gaussian nominal distribution. Then, it is shown that over AWGN channel, there exist an encoder and decoder that guarantee mean square uniform reconstruction.

Rate Distortion for a Class of Sources. Consider a class of sources which produce orthogonal zero mean encoder output processes $\{K_t \in \Re^d; t \in \mathbf{N}_+\}$, described via the following relative entropy constraint

$$\mathcal{D}_{SU}(g_{K^{T-1}}) = \{f_{K^{T-1}}; \frac{1}{T} H(f_{K^{T-1}}||g_{K^{T-1}}) \leq R_c + E_{f_{K^{T-1}}}[\frac{1}{2T}\sum_{t=0}^{T-1} K_t^{'} M_t K_t]\} \quad (8)$$

where the nominal density function $g_{K^{T-1}}$ is Gaussian distributed. That is, $g_{K^{T-1}} \sim N(0, diag\{\Lambda_0, \Lambda_1, ..., \Lambda_{T-1}\})$. The rate distortion for the class is defined by the minimax problem

$$R_{T,r}(D_v) \stackrel{\triangle}{=} \inf_{P(d\tilde{K}^{T-1};k^{T-1}) \in \mathcal{M}_{DC}} \sup_{f_{K^{T-1}} \in \mathcal{D}_{SU}(g_{K^{T-1}})} I(K^{T-1}; \tilde{K}^{T-1}), \quad (9)$$

where $I(.;.)$ is the mutual information [17] and $\mathcal{M}_{DC} \stackrel{\triangle}{=} \{P(d\tilde{K}^{T-1}; k^{T-1}); \frac{1}{T}\sum_{t=0}^{T-1} E||K_t - \tilde{K}_t||^2 \leq D_v\}$.

If we work on the space of probability measures induced by the densities, then it can be shown that two constraint sets are compact, and hence the problem is equivalent to

$$R_T^{sup}(D_v) = \sup_{f_{K^{T-1}} \in \mathcal{D}_{SU}(g_{K^{T-1}})} \inf_{P(d\tilde{K}^{T-1};k^{T-1}) \in \mathcal{M}_{DC}} I(K^{T-1}; \tilde{K}^{T-1}) \quad (10)$$

It can be further shown that the maximizing set can be restricted to orthogonal processes which are Gaussian.

For simplicity in analyzing, we consider the case of $K_t \in \Re$. The vector case is treated similarly.

Under assumption of $\bar{\sigma}(\Lambda_t M_t) < 1$, $\forall t \in \{0, 1, 2, ..., T-1\}$, where $\bar{\sigma}(.)$ denotes the biggest singular value, we have

$$R_T^{sup}(D_v) = \sum_{t=0}^{T-1} \frac{1}{2} \log \frac{\Psi_t^*}{D_v}, \; D_v < \min_{t \in \{0,1,...,T-1\}} \Psi_t^*$$
$$\Psi_t^* = \frac{1+s^*}{s^*} \frac{\Lambda_t}{1 - M_t \Lambda_t} \quad (11)$$

where $s^* > 0$ is the unique solution to the following equation

$$-\frac{1}{2} \log \frac{1+s^*}{s^*} + \frac{1}{s^*} + \frac{1}{2T} \sum_{t=0}^{T-1} \log(1 - \Lambda_t M_t) = R_c. \quad (12)$$

Computation of Robust Shannon Lower Bound. It can be easily shown that when $\lim_{T\to\infty} \Lambda_T = \Lambda_\infty$ and $\lim_{T\to\infty} M_T = M$, the robust Shannon lower bound is given by

$$R_{S,r}(D_v) = \frac{1}{2} \log \frac{\frac{1+s^*}{s^*} \frac{\Lambda_\infty}{1-M\Lambda_\infty}}{D_v} \quad (13)$$

Thus, for $D_v < \min_{t \in \mathbf{N}_+} \frac{1+s^*}{s^*} \frac{\Lambda_t}{1-M\Lambda_t}$, under assumptions of $\lim_{T\to\infty} M_T = M$, and $\lim_{T\to\infty} \Lambda_T = \Lambda_\infty$, the robust Shannon lower bound is an exact approximation of $R^{sup}(D_v) \stackrel{\triangle}{=} \lim_{T\to\infty} \frac{1}{T} R_T^{sup}(D_v)$. That is, $R^{sup}(D_v) = R_{S,r}(D_v) = \mathcal{H}_r(\mathcal{K}) - \frac{1}{2}\log(2\pi e D_v)$.

Realization of a Communication Link Matched to the Uncertain Source. Next, consider the following AWGN channel

$$\tilde{Z}_t = Z_t + \tilde{W}_t, \quad \tilde{W}_t \text{ orthogonal} \sim N(0, W_c),$$
$$Z_t \in \Re, \; E(Z_t^2) \leq P_t, \quad (14)$$

where $\tilde{W}_t$ is independent of Z_t, $\forall t \in \mathbf{N}_+$.

Under assumptions of $\lim_{T\to\infty} M_T = M$ and $\lim_{T\to\infty} \Lambda_T = \Lambda_\infty$, it can be shown that if the encoder multiplies K_t by $\alpha_t = \sqrt{\frac{\beta_t W_c}{D_v}}$ ($D_v < \min_{t \in \mathbf{N}_+} \frac{1+s^*}{s^*} \frac{\Lambda_t}{1-M_t\Lambda_t}$) ,i.e., $Z_t = \alpha_t K_t$, and transmits it under transmission power constraint $E(Z_t^2) = \alpha_t^2 E(K_t^2) \leq \frac{\beta_t W_c}{D_v} \frac{1+\eta_t^*}{\eta_t^*} \frac{\Lambda_t}{1-M_t\Lambda_t} \stackrel{\triangle}{=} P_t$, where $\eta_t^* > 0$ is the unique solution of the following equation $-\frac{1}{2}\log\frac{1+\eta_t^*}{\eta_t^*} + \frac{1}{\eta_t^*} + \frac{1}{2}\log(1 - \Lambda_t M_t) = R_c$, we have $\mathcal{C} = R^{sup}(D_v) = R_{S,r}(D_v)$. On the other hand, if the decoder multiplies the channel outputs by $\gamma_t = \sqrt{\frac{D_v \beta_t}{W_c}}$ to produce $\tilde{K}_t = \gamma_t \tilde{Z}_t$, for $D_v < \min_{t\in\mathbf{N}_+} \frac{1+s^*}{s^*}\frac{\Lambda_t}{1-M_t\Lambda_t}$, we have an end to end transmission with distortion

$$\begin{aligned}
E(K_t - \tilde{K}_t)^2 &= E(K_t - \gamma_t\alpha_t K_t - \gamma_t \tilde{W}_t)^2 \\
&= (1-\beta_t)^2 E(K_t^2) + \gamma_t^2 E(\tilde{W}_t^2) \\
&\leq (1-\beta_t)^2 \frac{1+\eta_t^*}{\eta_t^*}\frac{\Lambda_t}{1-M_t\Lambda_t} \\
&\quad +\gamma_t^2 E(\tilde{W}_t^2) \\
&= (1-1+\frac{D_v}{\frac{1+s^*}{s^*}\frac{\Lambda_t}{1-M_t\Lambda_t}})^2 \\
&\quad .\frac{1+\eta_t^*}{\eta_t^*}\frac{\Lambda_t}{1-M_t\Lambda_t} + \beta_t D_v \\
&= \frac{D_v^2}{(\frac{1+s^*}{s^*})^2\frac{\Lambda_t}{1-M_t\Lambda_t}}\frac{1+\eta_t^*}{\eta_t^*} + D_v \\
&\quad - \frac{D_v^2}{\frac{1+s^*}{s^*}\frac{\Lambda_t}{1-M_t\Lambda_t}} \\
&\quad \forall f_{K^{T-1}} \in \mathcal{D}_{SU}(g_{K^{T-1}}).
\end{aligned} \tag{15}$$

Thus, under assumption of $\lim_{T\to\infty} M_T = M$ and $\lim_{T\to\infty}\Lambda_T = \Lambda_\infty$, for $D_v < \min_{t\in\mathbf{N}_+}\frac{1+s^*}{s^*}\frac{\Lambda_t}{1-M_t\Lambda_t}$ using the proposed encoding scheme a uniform mean square observability in the form

$$\limsup_{T\to\infty} \sup_{f_{K^{T-1}}\in\mathcal{D}_{SU}(g_{K^{T-1}})} \frac{1}{T}\sum_{t=0}^{T-1} E(K_t - \tilde{K}_t)^2 \leq D_v \tag{16}$$

is obtained over the AWGN channel (14) with capacity $R_{S,r}(D_v)$.

The Vector Case $K_t \in \Re^d$**.** Extension of the above results to the vector case, i.e., to the case where an uncertain source produces orthogonal zero mean process $\{K_t\in\Re^d; t\in\mathbf{N}_+\}$, is straightforward. For the vector case

$$R_T^{sup}(D_v) = \sum_{t=0}^{T-1}\frac{1}{2}\log\frac{d\det\Sigma_t^*}{D_v}, \quad \frac{D_v}{d} < \min_i \lambda_{ti}^*, \forall t \tag{17}$$

where $i = \{1,2,...,d\}$,

$$\Sigma_t^* = diag\{\lambda_{t1}^*,...,\lambda_{td}^*\} \tag{18}$$

and

$$\Psi_t^* = \frac{1+s^*}{s^*}\Lambda_t(I_d - \Lambda_t M_t)^{-1} = E_t^*\Sigma_t^* E_t^{*'}, \tag{19}$$

in which $\bar{\sigma}(\Lambda_t M_t) < 1$, E_t^* is the unitary matrix that diagonalize Ψ_t^* and $s^* > 0$ is the unique solution of the following equation

$$\begin{aligned}
&-\frac{d}{2}\log\frac{1+s^*}{s^*} - \frac{d}{2} + \frac{1}{2T}\sum_{t=0}^{T-1}\log\det(I_d - \Lambda_t M_t) \\
&+\frac{1}{2T}\frac{1+s^*}{s^*}\sum_{t=0}^{T-1} trac[(I_d - M_t\Lambda_t)^{-1}(I_d-\Lambda_t M_t)] \\
&= R_c.
\end{aligned} \tag{20}$$

Next, consider the following AWGN channel

$$\begin{aligned}
&\tilde{Z}_t = Z_t + \tilde{W}_t, \quad \tilde{W}_t \text{ orthogonal} \sim N(0, W_c), \\
&W_c = diag\{W_1,...,W_d\}, \\
&Z_t = [Z_{t1} \ ... \ Z_{td}]' \in \Re^d, \quad E(Z_{ti}^2) \leq P_{ti}, \quad 1\leq i\leq d
\end{aligned} \tag{21}$$

It can be shown that when $\lim_{T\to\infty} M_T = M$ and $\lim_{T\to\infty}\Lambda_T = \Lambda_\infty$, uniform mean square observability of such uncertain source over the AWGN channel (21) is obtained by transmitting $\mathcal{C} = R^{sup}(D_v)$ ($= R_{S,r}(D_v)$ for sufficiently small D_v) if the encoder and decoder are defined as follows. The encoder multiplies $\bar{K}_t = E_t^{*'} K_t$ by

$$\mathcal{A}_t = diag\{\sqrt{\frac{\eta_{t1}W_1}{\frac{D_v}{d}}},...,\sqrt{\frac{\eta_{td}W_d}{\frac{D_v}{d}}}\}, \tag{22}$$

where $\eta_{ti} = 1 - \frac{\frac{D_v}{d}}{\lambda_{ti}^*}$ and $\frac{D_v}{d} \leq \min_i \lambda_{ti}^*$, $\forall t$ and transmits $Z_t = \mathcal{A}_t\bar{K}_t$ under power constraint $E(Z_{ti}^2) \leq \frac{\eta_{ti}W_i}{\frac{D_v}{d}}\lambda_{ti}^* \triangleq P_{ti}$, $1\leq i\leq d$. On the other hand, the decoder multiplies the channel outputs by

$$\mathcal{B}_t = diag\{\sqrt{\frac{D_v}{d}\frac{\eta_{t1}}{W_1}},...,\sqrt{\frac{D_v}{d}\frac{\eta_{td}}{W_d}}\} \tag{23}$$

to produce $\bar{\tilde{K}}_t = \mathcal{B}_t\tilde{Z}_t$ and subsequently, $\tilde{K}_t = E_t^*\bar{\tilde{K}}_t$.

Remark 3.1: From the results of this section and Theorem 2.4, it is concluded that for a given distortion value D_v (sufficiently small), $\mathcal{C} = R_{S,r}(D_v)$ is the minimum capacity under which there exists an encoding scheme for uniform mean square reliable data reconstruction of the process $\{K_t; t\in\mathbf{N}_+\}$, in which this capacity is achieved by choosing $\mathcal{A}_t$ and $\mathcal{B}_t$, as described by (22) and (23), respectively.

IV. Uncertain Fully Observed Controlled Gauss Markov System.

Consider the control/communication system of Fig. 1 described by the following AWGN channel

$$\begin{aligned}
&\tilde{Z}_t = Z_t + \tilde{W}_t, \ \tilde{W}_t \text{ orthogonal} \sim N(0,W_c), \\
&W_c = diag\{W_1,...,W_d\}, \quad Z_t = [Z_{t1} \ ...Z_{td}]', \\
&E(Z_{ti}^2) \leq P_{ti}, \qquad i = 1,2,...,d,
\end{aligned} \tag{24}$$

and the following uncertain system

$$(\Omega, \mathcal{F}(\Omega), P; \{\mathcal{F}_t\}_{t\geq 0}): \begin{cases} X_{t+1} = AX_t + NU_t + BW_t + B\bar{W}_t, & X_0 = X, \\ Y_t = H_t, & H_t = X_t \end{cases} \tag{25}$$

where $X_t \in \Re^d$, $U_t \in \Re^o$, $W_t \in \Re^m$, $\bar{W}_t \in \Re^m$, $X_0 \sim N(\bar{x}_0, \bar{V}_0)$, $H_t \in \Re^d$ is the signal to be controlled, W_t is i.i.d. $\sim N(0,\Sigma_W), \Sigma_W > 0$, $\bar{W}_t$ is the perturbed noise random process which is $\{\sigma\{W_l\}; l\leq t-1\}$ adapted, and $\{X_0, W_t, \bar{W}_t\}$ are mutually independent.

The nominal system associated with the uncertain system (25) is the following fully observed system

$$(\Omega, \mathcal{F}(\Omega), \Pi; \{\mathcal{F}_t\}_{t\geq 0}): \begin{cases} X_{t+1} = AX_t + NU_t + BW_t, & X_0 = X, \\ Y_t = H_t, & H_t = X_t \end{cases} \tag{26}$$

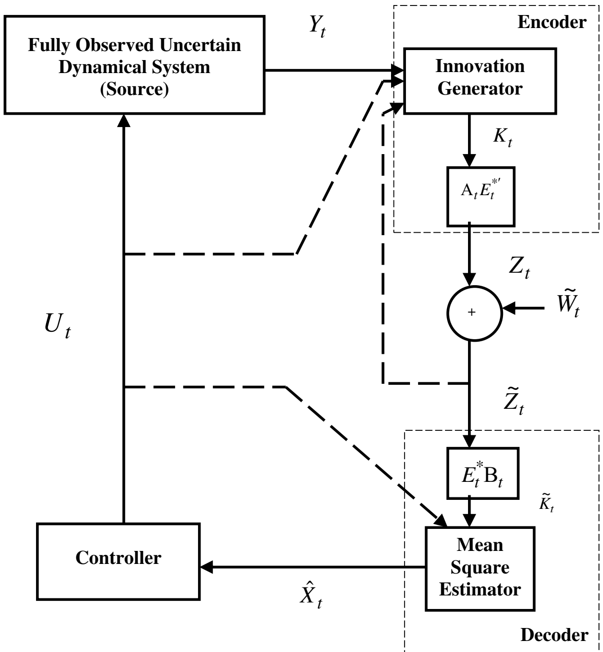

Fig. 2. Control/communication system subject to uncertainty in the source

The uncertain system (25) is subject to the following sum quadratic uncertainty constraint $\{\{\bar{W}_t\}_{t=0}^{T-2}; g(\bar{W}^{T-2}) \leq 0\}$; $g(\bar{W}^{T-2}) \triangleq E_P[\frac{1}{2T}\Big(\sum_{t=0}^{T-2}(\bar{W}_t^{'}\Sigma_W^{-1}\bar{W}_t - \sum_{t=0}^{T-1} X_t^{'}MX_t\Big) - R_c]$.
For $P(dY^{T-1}) = f_{Y^{T-1}}dY^{T-1}$ and $\Pi(dY^{T-1}) = g_{Y^{T-1}}dY^{T-1}$ associated with the uncertain and nominal systems (25) and (26), respectively, consider the following robust entropy rate problem.

$$\frac{1}{T}H_r(f^*_{Y^{T-1}}) = \sup_{f_{Y^{T-1}}\in\mathcal{D}_{SU}(g_{Y^{T-1}})}\frac{1}{T}H_S(f_{Y^{T-1}}) \quad (27)$$

where $H_s(.)$ is the Shannon differential entropy [17] and

$$\mathcal{D}_{SU}(g_{Y^{T-1}}) = \{f_{Y^{T-1}}; P(dY^{T-1}) = f_{Y^{T-1}}dY^{T-1},$$
$$\Pi(dY^{T-1}) = g_{Y^{T-1}}dY^{T-1}, \frac{1}{T}H(P||\Pi) \leq R_c$$
$$+E_P[\frac{1}{2T}\sum_{t=0}^{T-1}Y_t^{'}MY_t]\}. \quad (28)$$

Suppose $B^{'}(B\Sigma_W B^{'})^{-1}B \leq \Sigma_W^{-1}$ and consider the following unconstraint problem (here we assume existence of solution yielding a finite entropy).

$$\frac{1}{T}H_r(f^{*,s^*}_{Y^{T-1}}) = \min_{s\geq 0}\sup_{\{\bar{W}_t\}_{t=0}^{T-2}}\Big\{-\frac{1+s}{2T}E_P[\sum_{t=0}^{T-2}\bar{W}_t^{'}\Sigma_W^{-1}$$
$$.\bar{W}_t] - \frac{1}{T}E_P[\log g_{Y^{T-1}}] + sR_c$$
$$+\frac{s}{2T}E_P[\sum_{t=0}^{T-1}Y_t^{'}MY_t]\Big\}. \quad (29)$$

Subsequently, the robust entropy rate is given by

$$\mathcal{H}_r(\mathcal{Y}) = \lim_{T\to\infty}\frac{1}{T}H_r(f^{*,s^*}_{Y^{T-1}}). \quad (30)$$

Next, following the stochastic dynamic programming [22], the solution to the robust entropy problem (29) is given in the following Theorem.

Theorem 4.1: Consider the robust entropy problem (29) and (30). Let $B^{'}(B\Sigma_W B^{'})^{-1}B < (1+s)\Sigma_W^{-1}$ for some $s \geq 0$. Then,
i)

$$\bar{W}_t^* = -[B^{'}(B\Sigma_W B^{'})^{-1}B - (1+s)\Sigma_W^{-1}$$
$$+B^{'}\Xi_{t+1}B]^{-1}B^{'}\Xi_{t+1}AX_t \quad (31)$$

where Ξ_t is a real symmetric solution of

$$\Xi_t = A^{'}\Xi_{t+1}A - A^{'}\Xi_{t+1}B[B^{'}(B\Sigma_W B^{'})^{-1}B$$
$$-(1+s)\Sigma_W^{-1} + B^{'}\Xi_{t+1}B]^{-1}B^{'}\Xi_{t+1}A + sM$$
$$\Xi_{T-1} = sM. \quad (32)$$

and $s \geq 0$ is the minimizing solution of the following equation

$$Z(s^*) = \min_{s\geq 0}\{sR_c + \frac{1}{2T}trac(\Xi_0\bar{V}_0)$$
$$+\frac{1}{2T}\sum_{t=1}^{T-1}trac(B^{'}\Xi_t B\Sigma_W)\} \quad (33)$$

ii) If (A,B) is controllable, A and $B^{'}(B\Sigma_W B^{'})^{-1}B - (1+s)\Sigma_W^{-1}$ are invertible, and $\beta(\eta) > 0$ for some η; $|\eta| = 1$ where $\beta(\eta)$ is the rational matrix function given by

$$\beta(\eta) = B^{'}(B\Sigma_W B^{'})^{-1}B - (1+s)\Sigma_W^{-1} + B^{'}(\eta^{-1}I_d$$
$$-A^{'})sM(\eta I_d - A)^{-1}B, \quad s \geq 0. \quad (34)$$

Then

$$\mathcal{H}_r(\mathcal{Y}) = sR_c + \frac{q}{2}\log(2\pi e) + \frac{1}{2}\log\det(B\Sigma_W B^{'})$$
$$+\min_{s\geq 0}\{sR_c + \frac{1}{2}trac(B^{'}\Xi_\infty B\Sigma_W)\} \quad (35)$$

where Ξ_∞ is the solution of the following Algebraic Riccati equation appearing in the H^∞ estimation and control problems

$$\Xi_\infty = A^{'}\Xi_\infty A - A^{'}\Xi_\infty B[B^{'}(B\Sigma_W B^{'})^{-1}B - (1+s)$$
$$.\Sigma_W^{-1} + B^{'}\Xi_\infty B]^{-1}B^{'}\Xi_\infty A + sM. \quad (36)$$

Next, let $U_t \in \mathcal{U}_t \triangleq \{U_t : \Re^{t(o+q)} \to \Re^o; U_t \in \mathcal{G}_{t-1}^U\}$; $\mathcal{G}_t^U \triangleq \sigma\{Y_0,...,Y_t; U_0,...,U_t\}$. The objective is to design an encoder, decoder and controller for mean square stability subject to the following cost functional.

$$\lim_{T\to\infty}\inf_{U^{T-1}\in\mathcal{U}_0\times\mathcal{U}_1\times...\times\mathcal{U}_{T-1}}\sup_{\{\{\bar{W}_t\}_{t=0}^{T-2}; g(\bar{W}^{T-2})\leq 0\}}$$
$$\frac{1}{T}J(X_{0,T-1}, U_{0,T-1}) \quad (37)$$

where $J(X_{0,T-1}, U_{0,T-1}) = \frac{1}{2} E_P \sum_{t=0}^{T-1} (||X_t||^2 + ||U_t||_H^2), \quad (H > 0)$.

Fig. 2 illustrates encoding and stabilizing schemes for uniform observability and robust stability of the uncertain system (25) subject to the cost functional (37). The encoder, decoder and controller will be an extension of the results of Section III and the results of [11] and [23]; and hence we omit the detail of the design of the encoding and stabilizing schemes due to the space limitation.

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