The Element-Gain Paradox for a Phased-Array Antenna

PETER W. HANNAN, SENIOR MEMBER, IEEE

Summary—In a phased-array antenna with a very large number of regularly-spaced radiating elements, the gain realized at the beam peak is equal to the number of elements times the gain realized in the same direction when only one typical element is excited. The ideal radiation pattern of one such element in a large planar array has a cosine variation of gain with angle when the elements are closely spaced, and has a peak value of gain equal to $4\pi A/\lambda^2$ where A is the area allotted to each element.

The active impedance of each element in a practical phased array varies with scan angle, because of mutual coupling between the elements. The associated mismatch causes power to be returned to the generators, thereby reducing the gain realized by the array and by the element. The element pattern, measured in the proper environment of surrounding elements, deviates from the ideal pattern in proportion to this effect.

Mutual coupling is inherently unavoidable in a closely-spaced infinite array of elements; for example, in a square array with less than $\lambda/\sqrt{2}$ spacing. There is a loss of element efficiency caused by the coupling, and since coupling increases with closer spacing, this accounts for the lower gain expected from ideal elements with reduced allotted area. Grating lobes can exist when the elements are not closely spaced; for example, in a square array with more than $\lambda/2$ spacing. In this case, the ideal pattern is truncated to discriminate against grating lobes; this gives the higher gain expected from ideal elements with larger allotted area.

It is concluded that in a phased-array antenna having a very large number of regularly-spaced radiating elements, perfect impedance match for all scan angles can be postulated for every typical element without encountering any real discrepancy in the determination of element gain. In the absence of grating lobes, such an antenna would realize the greatest possible gain for all scan angles.

I. Introduction

About two years ago this writer ran into an apparent paradox involving the gain of a radiating element in a phased-array antenna. While studying this problem, several interesting concepts and relations governing the behavior of an array element were generated; these concepts are the principal contribution of this paper. However the original problem has undoubtedly been discovered by others working in the field of array antennas; one such instance has been communicated [1] to the writer. This paper, therefore, will include an outline of the original difficulties, in the expectation that they will be of interest to the reader.

II. Preliminary View of the Element-Gain Pattern for an Infinite Array

An element pattern of an array antenna is the radiation pattern obtained when only one element is excited by a generator, as indicated in Fig. 1. The gain of this pattern is a factor in determining the gain of the array when all the elements are excited; therefore, it is of interest to know the gain as well as the shape of the element pattern. Both of these quantities are contained in a plot of element gain $g(\theta, \phi)$ as a function of angle. This plot will be termed the *element-gain pattern*;[1] a possible one is indicated in Fig. 1.

When the antenna array contains a very large number of regularly-spaced elements, the behavior of every element is the same except for those relatively few non-typical elements near the edge. The essential properties of such an array can be conveniently determined on the basis of an array having an infinite number of elements [2]. Fig. 2 indicates a portion of an *infinite planar array*, in which the elements happen to have a square arrangement.

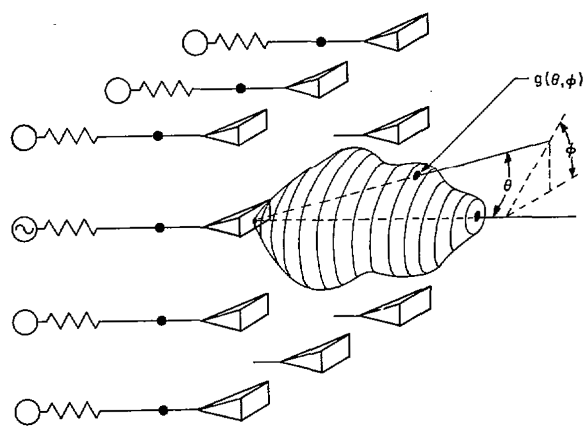


Fig. 1—The element-gain pattern.

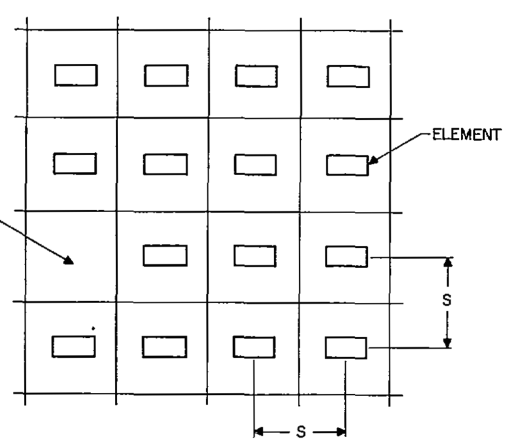


Fig. 2—An infinite planar array.

Manuscript received August 19, 1963; revised December 9, 1963. This paper was prepared in the course of antenna development for the Bell Telephone Laboratories on a project for the U. S. Army.
The author is with Wheeler Laboratories, Inc., Smithtown, N. Y.

[1] The element-gain pattern is, of course, a plot in terms of power ratio, not voltage ratio. The reference is an isotropic radiator, which has a gain of unity.

An area A is allotted to each element in the infinite array. This is the maximum area available to each element, and is usually greater than the physical size of the actual element. It is natural to assume that the maximum gain obtainable from an element in the array is related to the area A by the well-known gain formula for apertures large compared with a wavelength, because the entire array is indeed large. Furthermore, since the effective area of an element should be proportional to its projected area in the direction of interest, the element gain should have a cos θ variation with angle. Based on this intuitive reasoning, the maximum element gain would be

$$g_{\max}(\theta, \phi) = \frac{4\pi A}{\lambda^2} \cos \theta. \tag{1}$$

This relation gives a fundamental upper limit to the gain obtainable in an element of an infinite planar array. It also implies that the ideal shape of the gain pattern of such an element would approach the cos θ variation. However, there are factors not contained in (1) which must be considered in any objective analysis of element gain.

III. Effect of Reflection on the Element-Gain Pattern

In an infinite phased-array antenna the radiation pattern has an infinitesimal width, and it is convenient to consider the antenna gain only in the direction where all the elements add in phase; this corresponds to the peak of the antenna pattern. When all the elements have identical patterns and are excited with equal amplitudes, as is the case in an infinite array, the principle of superposition may be combined with consideration of the power available from the generators to yield the familiar relation

$$G_r(\theta, \phi) = n g_r(\theta, \phi), \tag{2}$$

where $G_r(\theta, \phi)$ is the gain realized by the array antenna in the direction θ, ϕ when the elements are excited to add in phase in that direction, $g_r(\theta, \phi)$ is the gain realized in the same direction when only one element is excited, and n is the number of elements (infinite in this case). It is appropriate to remind the reader that the element gain must be determined with the element located in its actual environment of surrounding elements, and that every element must be connected to the same impedance when determining the element gain and the array gain.[2]

The gain realized by an antenna system is always less than the directive gain [5] of the antenna alone. In addition to dissipation, there may be an impedance mismatch which returns power to the transmitter. When these two losses are included, the resulting value for gain will be termed the *realized gain*. It is important to recognize that (2) applies to realized gain, not directive gain. (Also, the formula does not apply to power gain [5], which term is generally understood to exclude loss caused by impedance mismatch.) The subscript r for the two gains in (2) denotes the term "realized."

It will be assumed throughout this paper that the radiating elements and their associated transmission lines have no internal dissipation. In this case, the difference between the directive antenna gain and the realized antenna gain is caused entirely by power which is returned to the generators. Thus the ratio of realized antenna gain $G_r(\theta, \phi)$ in some direction to directive gain $G_d(\theta, \phi)$ in the same direction is equal to

$$\frac{G_r(\theta, \phi)}{G_d(\theta, \phi)} = 1 - |R(\alpha, \beta)|^2, \tag{3}$$

where $|R(\alpha, \beta)|^2$ is the ratio of returned power to available power, and α, β are the phases between excitation of adjacent elements in the rows and columns which produce radiation by the infinite array in the θ, ϕ direction.

The returned power is caused by departure from conjugate match between the element impedance and the generator impedance. The element impedance of significance in this case is the *active impedance* of the element, that is, the impedance measured while all the other elements are excited with the appropriate phase. As will be discussed later, mutual coupling between the elements causes the active impedance and the returned power to vary with excitation phase; hence R in (3) is a function of α and β.

If the generators happen to be matched to the transmission lines leading to the elements, R is also the *voltage reflection coefficient* in these transmission lines. It is an *active* reflection coefficient for the reasons just mentioned. In the general case of generators that are not matched to the element lines, R is not a true reflection coefficient in any physical transmission line. However, R may still be considered as a voltage reflection coefficient where this reflection is taken to exist within a hypothetical generator. Fig. 3 indicates this generator consisting of a voltage source and internal resistance, followed by a hypothetical transmission line matched to the internal resistance, and ending in a mismatched connection (shown as a dot) to the actual element transmission line. The "*equivalent* reflection coefficient" is considered to exist within the hypothetical matched transmission line. It should also be mentioned that the interpretation of returned power in terms of a voltage reflection coefficient or an impedance mismatch applies only when a *single polarization* is involved. When two polarizations can propagate in the transmission lines, as in an array system designed for right- and left-hand circular polarization, the returned power may be di-

[2] Historically, (2) has been applied to the case of an *isolated* element which is transferred to an array of many elements. In that application it is necessary to assume that mutual coupling and interaction effects are negligible; unfortunately this is usually far removed from reality. Only recently have these considerations been clearly stated [3], [4].

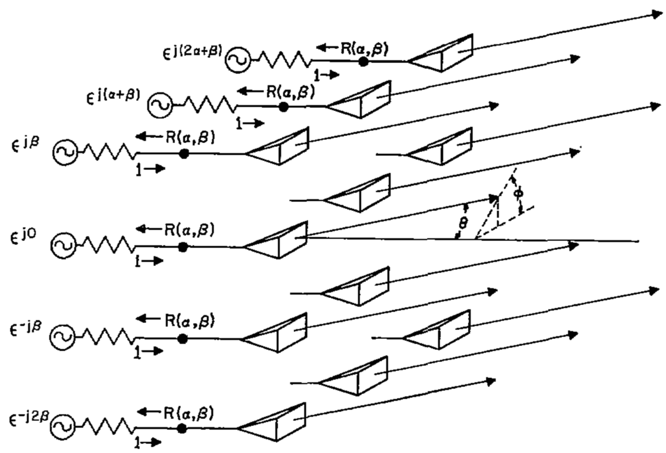


Fig. 3—Reflection of a scanning array.

vided between the two. Although this case can also be analyzed in relation to the element-gain pattern, it is beyond the scope of this paper to perform such an analysis.

The peak directive gain of a large, uniformly-excited planar array which has no grating lobes is related to the area of the array aperture by the standard formula for apertures large compared with a wavelength. Thus for the infinite array,[3] the directive gain is

$$G_{do}(\theta, \phi) = \frac{4\pi n A}{\lambda^2} \cos\theta, \tag{4}$$

where θ, ϕ is again the direction in which all the elements add in phase, and the subscript o means that no grating lobes are permitted.

Now combining (2), (3), and (4) yields for the element realized gain

$$g_{ro}(\theta, \phi) = \frac{4\pi A}{\lambda^2} \cos\theta(1 - | R(\alpha, \beta) |^2). \tag{5}$$

It should be emphasized that (5) relates *two conditions of operation* of the array. The element realized gain is obtained by exciting only one element of the array and measuring gain in the θ, ϕ direction, as indicated in Fig. 1. The reflection coefficient is determined by exciting every element with a progressive phase shift α, β such that a beam in the θ, ϕ direction is obtained, and measuring the ratio of returned to available power. This latter condition is, of course, the operational condition, and is indicated in Fig. 3.

Comparing (5) with the simple intuitive equation (1), it is seen that the two differ only by the factor involving the reflection coefficient. Perhaps (1) is the more fundamental relationship because it involves only the antenna, and not the interaction between the antenna and the generators which theoretically can always be tuned out. However, in a scanning array antenna it is not customary to tune out the mismatch as a function of scan; hence, in practice, it is the realized gain that is significant. The element realized gain of (5) determines the array realized gain, according to (2).

In a typical array antenna the reflection coefficient would vary with scan angle. This would modify the shape of the element realized-gain pattern from the simple cos θ shape, as has been recognized by others [4], [6]. Since the reflection coefficient in (5) is actually an equivalent reflection coefficient which accounts for the power returned to the generators, its value depends partly on the impedance of the generators connected to the elements. Therefore the *element realized-gain pattern depends on the impedance of the generators.* Although this dependence appears to conflict with the usual rules of antenna pattern measurement, it should be pointed out that only the impedance of the inactive generators surrounding the excited element affects the element pattern shape, while the impedance of the active generator affects only the level of the element-gain pattern.

The array antenna system which has been discussed is one in which each element of the operating array is connected to a separate generator. This is often called an *active array.* When a single generator excites all the elements through a network of power dividers and phase shifters, the antenna may be called a *passive array.* Such an antenna does not have the obvious identical performance for every element that the infinite active array has. Furthermore, reflections from the elements do not necessarily return to the generator, but may be absorbed in terminations or reradiated in other directions [7], [8], depending on the particular network used. Thus the passive array is actually more complicated to analyze than the active one. The active array not only is relatively simple, but also it provides the concepts and fundamental limitations for any array; thus the active array is considered exclusively in this paper.

[3] With the infinite array, it is not necessary to consider end-fire gain because the projected aperture is large compared with the end-fire aperture for all values of θ less than 90°.

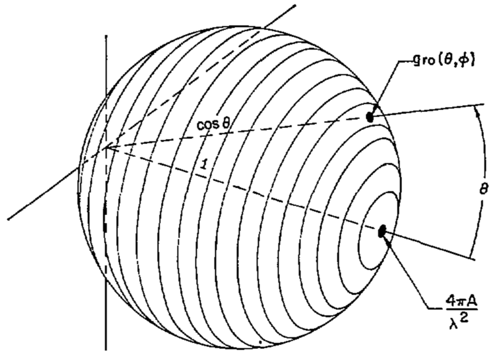


Fig. 4—The ideal element realized-gain pattern when no grating lobes exist.

It is interesting to recognize that, while the relations in this paper are derived for a *transmitting* array, they apply as well to a *receiving* array in which there is a receiver connected to each element. In this case, the equivalent reflection coefficient is that quantity which accounts for the power reradiated by the antenna system because of a departure from conjugate match between the receiver impedance and the element impedance. Although the element is receiving, it is still the *active impedance* of the element in *transmission* which is significant here. Thus the relations in this paper obey the usual laws of reciprocity, even though the antenna is an active array rather than a passive one.

IV. The Paradox

Suppose that it is possible to obtain an array antenna system which has perfect impedance match for all scan angles. Then in (5) the reflection coefficient R is zero over the range of α and β corresponding to $0<\theta<\pi/2$ and $0<\phi<2\pi$. The element realized-gain pattern then becomes the ideal element pattern mentioned in Section II. This pattern, shown in Fig. 4, has a peak realized gain of $4\pi A/\lambda^2$ and a $\cos\theta$ shape.

If all the elements of the array are perfectly matched for all directions of array radiation, it might seem reasonable to assume that when a single element is excited it will also be matched. If it is also assumed that the elements have no dissipation, then the element realized gain is equal to the element directive gain. Now a gain pattern having the $\cos\theta$ shape described above has a directivity[4] of four [9]. Therefore the peak value of element realized gain should also be four. This, however, does not in general correspond to the value of $4\pi A/\lambda^2$ derived above.

To illustrate this discrepancy, consider an array with a square arrangement of elements. If the element spacing is $\lambda/2$, then $4\pi A/\lambda^2$ equals π instead of four. If the element spacing is $\lambda/\sqrt{2}$, then $4\pi A/\lambda^2$ equals 2π instead of four. There appears to be a paradox here; evidently some of the assumptions made are not valid, or there are still other factors which must be considered. Of great interest is the question of *whether it is really possible to have an array antenna system which has perfect impedance match for all scan angles.* In the following sections of this paper, further analysis will be made to answer these questions.

V. Effect of Grating Lobes on the Element-Gain Pattern

Eqs. (4) and (5) were derived for the case of an array antenna in which no grating lobes were permitted. When grating lobes do exist, a different result is obtained.

Grating lobes [10] occur in an array antenna when the spacing between elements is great enough to permit all the elements to add in phase in one or more directions other than the direction of the main lobe. In an infinite array the scan-plane beamwidth of any of these in-phase lobes, although infinitesimal, is inversely proportional to the cosine of the angle of the lobe from broadside, because the projected aperture of the antenna varies by the cosine. Therefore the power radiated in one of these lobes is proportional to the directive gain in the lobe direction, divided by the cosine of the lobe angle. When many such lobes exist, the total power radiated is the sum of all the lobe powers. If this total power is set equal to the power radiated when only the main lobe is permitted to exist, the following relation is obtained:

$$\sum_{m=0}^{\max} \frac{G_d(\theta_m, \phi_m)}{\cos\theta_m} = \frac{G_{do}(\theta, \phi)}{\cos\theta} \tag{6}$$

where on the left, the subscript m identifies the lobe ($m=0$ for main lobe, $m=1$ for first grating lobe, etc.), and the summation covers all the lobes that exist. In all cases, θ and ϕ represent the directions in which all the elements add in phase.

Now the derivation of (4) relies on a determination of total power radiated. If the same derivation is made for the case which includes grating lobes, consideration of (6) yields the following relation:

$$\sum_{m=0}^{\max} \frac{G_d(\theta_m, \phi_m)}{\cos\theta_m} = \frac{4\pi n A}{\lambda^2}. \tag{7}$$

[4] Directivity is the peak value of directive gain [5].

Combining (7) with (2) and (3) now yields the general expression describing the element realized gain,

$$\sum_{m=0}^{\max} \frac{g_r(\theta_m, \phi_m)}{\cos \theta_m} = \frac{4\pi A}{\lambda^2}(1 - |R(\alpha, \beta)|^2). \tag{8}$$

This reduces to (5) when there are no grating lobes. However, when the element spacing is great enough to permit grating lobes, there is no longer the simple situation which led to the paradox described in the preceding section. Before analyzing this, however, another aspect will be considered.

VI. Effect of Mutual Coupling on Element Efficiency

When an array antenna has elements which are closedly spaced, there is usually an appreciable amount of interaction between the elements. This interaction may be separated into several components, such as backward coupling, forward coupling, and scattering. It is the *backward coupling* which is of interest here; this will be referred to as the mutual coupling.

Determination of the mutual coupling between two elements is accomplished by connecting a generator to one element and a receiver to the other. For the measurement to be significant, these two elements must, of course, be in the same environment of surrounding elements that exists in the complete operating array, and all the surrounding elements must be connected to terminations (or inactive generators). Furthermore, the impedance of the generator, receiver, and terminations must have the correct value. There are two values for this impedance which may be chosen. In one case, the impedance is equal to that of the transmission lines leading to the elements; this corresponds to a measurement of a coefficient of the scattering matrix. In the other case, the impedance is equal to that of the generators in the complete operating array. While the former case is generally simpler to measure, interpretation of its significance is necessary if the operating system has generators which are not matched to the element lines.[5] The latter will be employed in this paper because of its direct application to the array system performance.

Fig. 5 shows the measurement of mutual coupling coefficient, which compares the signal available at the generator with the signal received by the receiver. Both the amplitude and the phase are measured, yielding a complex coupling coefficient. As the receiver is connected to each element line in turn, a set of complex coefficients is obtained; included in the set is the "self-coupling" coefficient or passive reflection coefficient, determined by means of a directional or standing-wave measurement.

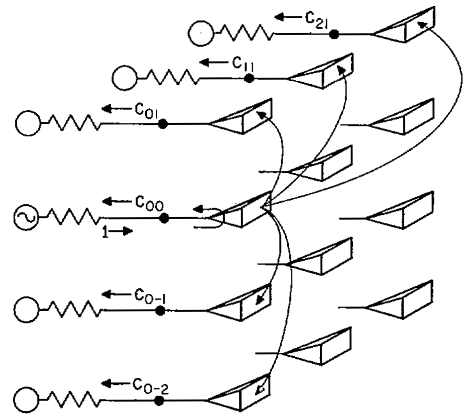


Fig. 5—Determination of mutual coupling.

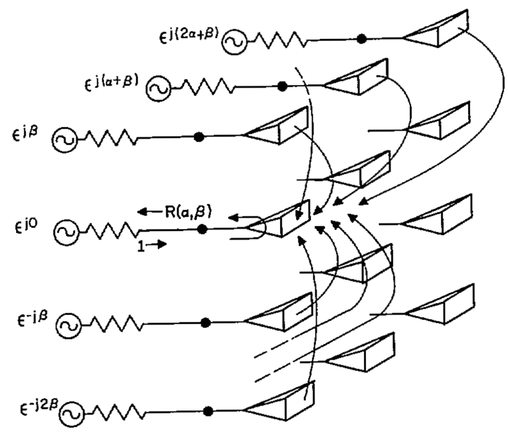


Fig. 6—Effect of mutual coupling on array reflection.

When the beam of the operating array antenna is scanned by varying the phase of element excitation, the presence of mutual coupling causes the signal reflected into each generator to vary. If all the coupling coefficients are available, the characteristic of reflection vs excitation phase can be computed. This is done simply by adding together the signals from all the generators as they are coupled back into one generator, as suggested in Fig. 6. Thus by reciprocity and superposition, the reflected signal is the sum of the products of the coupling coefficients times the appropriate generator phase factors. Each pair of elements which is symmetrical about the element in question contributes a term equal to $C_{pq} \exp j(p\alpha+q\beta)+C_{-p-q} \exp j(-p\alpha-q\beta)$ where p is the column and q is the row. Since $C_{pq}=C_{-p-q}$ because of symmetry, the contribution of each pair of elements is $C_{pq} \cos (p\alpha+q\beta)+C_{-p-q} \cos (-p\alpha-q\beta)$. Thus the complete reflection coefficient is

[5] In this case, the determination of returned power would involve two quantities. One quantity is the active impedance of an element calculated from the reflection coefficient determined from a summation of the scattering coefficients as measured with terminating impedances which are all matched to their transmission lines. The other quantity is the actual impedance of the generators in the operating array. Comparison of these two quantities on the basis of departure from conjugate impedance match at a common reference plane would be necessary to determine the fraction of available power which is returned to the generators in the operating array.

$$R(\alpha, \beta) = \sum_{p=-\infty}^{+\infty} \sum_{q=-\infty}^{+\infty} C_{pq} \cos (p\alpha + q\beta) \quad (9)$$

$$= \sum_{p=-\infty}^{+\infty} \sum_{q=-\infty}^{+\infty} C_{pq}[(\cos p\alpha)(\cos q\beta) - (\sin p\alpha)(\sin q\beta)], \quad (10)$$

where R has the same interpretation as that discussed in Section III. Eq. (10) is obtained from (9) by the standard trigonometric identity. When there is horizontal and vertical symmetry, as in the ordinary case of linear polarization parallel to a row or column, the sin terms in (10) drop out. As mentioned previously, all the coupling coefficients C_{pq} are complex, as is the reflection coefficient R.

It is evident from (9) or (10) that when any of the mutual (p or $q>0$) coupling coefficients are not zero, the reflection coefficient R will vary with the phasing α, β. This variation confirms the statements made in Section III, and has been generally recognized for many years.[6] Now it is desirable to have an array that has no change of complex reflection coefficient with scan angle, because an impedance-matching process at one scan angle will then provide a match at all angles. One is therefore inclined to state that the *ideal antenna array* is one in which there is *no mutual coupling*. This, however, is *not always true*, as will be seen in the next section of this paper. In preparation for this, a relation between the characteristic of reflection vs phasing and the total coupled power will now be derived.

Eq. (10) involves a doubly infinite series of cosine and sine terms having arguments of increasing multiples of the independent variables. This is just a two-dimensional Fourier series [14] whose coefficients are the coupling coefficients. Now it is a well-known property of a Fourier series that the average power in the complete "waveform" is equal to the sum of the powers in all the "harmonics." In relation to (10), this means that

$$\frac{1}{\pi^2} \int_{\alpha=0}^{\pi} \int_{\beta=0}^{\pi} | R(\alpha, \beta) |^2 d\alpha d\beta = \sum_{p=-\infty}^{+\infty} \sum_{q=-\infty}^{+\infty} | C_{pq} |^2. \quad (11)$$

The significance of (11) for the array antenna is that *the average power, over a 0 to π interval of phasing in both coordinates, returned to one generator in the complete operating array, is equal to the net power returned to all generators when only one is excited.* This presumes equal power available at every generator for both conditions, because the R and each C in formulas (10) and (11) are quantities which are normalized to unit-amplitude generators. Fig. 7 illustrates the principle; a typical determination of average returned power for the operating array is shown in Fig. 7(a), and the net returned power when one element is excited is indicated in Fig. 7(b).

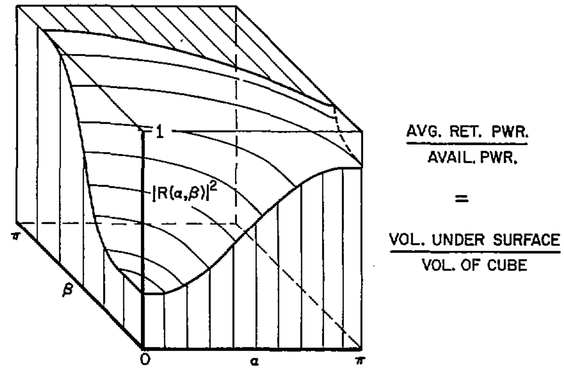
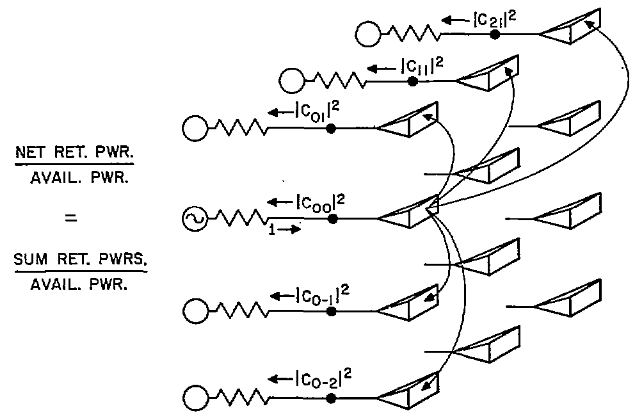


Fig. 7—Determination of average reflection and net coupling. (a) Characteristic of reflection vs phasing.



(b) View of coupled powers.

It should be mentioned that, although the principle was derived for a square or rectangular arrangement of elements in the array, it applies also to a triangular or any other regular arrangement, so long as the phasing coordinates are in orthogonal directions.

Consider now the significance of the net power returned to all the generators when only one is excited. This power is dissipated in the internal resistance of the generators, including the one active generator, and therefore does not radiate. Suppose that *the efficiency of an element in the array is defined as the power radiated divided by the power available, when only that element is excited.* This "element efficiency"[7] is then equal to one minus the ratio of the net power returned to the power available, and that ratio is equal to the right side of (11). Thus the element efficiency is

$$\text{element efficiency} = 1 - \frac{P_{\text{net ret}}}{P_{\text{avail}}} \quad (12)$$

$$= 1 - \sum_{p=-\infty}^{+\infty} \sum_{q=-\infty}^{+\infty} | C_{pq} |^2. \quad (13)$$

[6] In the past, the variation has usually been expressed in terms of active impedance [11], [12] rather than reflection coefficient. However, a variation of active impedance corresponds to the variation of returned power and equivalent reflection coefficient just described. Recently, the reflection coefficient (or VSWR) has been employed in at least several instances [3], [13].

[7] The "element efficiency" as defined above is not the same as the radiation efficiency [5] of the element, because the latter term does not include loss by reflection into the active generator. Only if the passive impedance of the element happened to be matched to the generator would the two types of efficiencies be equal.

By virtue of (11), the element efficiency can also be written in terms of the reflection characteristic of the operating array. Thus

$$\text{element efficiency} = 1 - \frac{1}{\pi^2}\int_{\alpha=0}^{\pi}\int_{\beta=0}^{\pi} |R(\alpha, \beta)|^2 d\alpha d\beta. \quad (14)$$

This result does not appear to be unreasonable, since it is seen that greatest element efficiency corresponds to the least average reflection in the operating array. In the following section, (14) will be employed to develop some ideal properties of an element in an array.

VII. The Ideal Element-Gain Pattern

In the early sections of this paper, the concept of an ideal element-gain pattern was introduced. This section will attempt to specifically define the ideal pattern, based on the results obtained in Sections V and VI. For simplicity of presentation, the discussion will be confined to a square arrangement of elements, and only two elements spacings will be considered, $\lambda/2$ and $\lambda/\sqrt{2}$. These two cases will serve to illustrate the principle.

A. Square Array, $\lambda/2$ Spacing

When the element spacing in a square array is $\lambda/2$ or less, no grating lobes can be formed, no matter what combination of excitation phases α, β are employed.[8] Hence, for this case, it is not necessary to refer to Section V. The results of Section VI, however, are significant for this case.

The ideal element of an array is one which somehow provides a reflection coefficient R equal to zero for all scan angles θ from 0° to 90°. This does *not* correspond to zero reflection coefficient for all *phasings* α, β, however, because for small element spacings there are some values of phasing where an infinite array cannot radiate. When the array cannot radiate, the reflection coefficient must be equal to unity.

The phasings corresponding to nonradiation by the array are those which exceed the values that yield a scan angle of 90°. Since the element spacing is such that no grating lobes can be formed, there is then no direction in which all the elements add in phase. Furthermore, in an *infinite array* the major lobe and the near minor lobes occupy an infinitesimal angular region; there is no residual radiation in any out-of-phase condition.

To determine the critical values of phasing which correspond to a 90° scan angle, it is convenient to employ the space-coordinate system based on the intersection of two cones [15], as indicated in Fig. 8 (next page). These cones are defined by the angles μ, ϵ, as shown; the sines of these angles are proportional to the phasing α, β, respectively, required to scan the beam to the cone intersection. Also, $\sin^2\mu+\sin^2\epsilon=\sin^2\theta$, so that a constant scan angle, variable plane of scan, plots as a circle on the $\sin\mu$, $\sin\epsilon$ plane [10]. The particular circle corresponding to the $\theta=90°$ boundary of radiation is shown in Fig. 9. In addition, the proportionality between the $\sin\mu$, $\sin\epsilon$ coordinates and the α, β phasing is shown in the general form dependent on element spacing S for the square array.

Inside the radiation circle the reflection coefficient R equals zero for an ideal array element; outside the circle R must be unity. Fig. 10 presents this result on the characteristic of reflection coefficient R vs phasing α, β, for $\lambda/2$ element spacing. With this particular spacing, it happens that the radiation circle is inscribed in that square of phasing described by α, β between $-\pi$ and π, so the ideal reflection-phasing characteristic has the particularly simple shape shown for this case. Nevertheless, it illustrates that there are some values of phasing which require complete reflection; for example, when α and β are *both* equal to π, the value of R must be unity.

It is interesting to consider the significance of this situation in terms of the mutual-coupling behavior. This ideal element provides a reflection which must vary with phasing. The Fourier series representing the reflection coefficient in (9) or (10) must therefore have mutual-coupling coefficients which are not zero, but have just the right values to provide the particular variation of reflection with phasing. Consequently, in the infinite array with $\lambda/2$ element spacing, *the ideal element must have mutual coupling.* In fact, even for a nonideal element in this array, that is, an element having a nonzero reflection coefficient in the radiating condition of phasing, there must be mutual coupling unless the reflection coefficient is unity for all phasings. Except for this trivial case, then, *mutual coupling must always be present* in the infinite square array with $\lambda/2$ element spacing.

Returning now to the information contained in Fig. 10, and applying it to (14), the element efficiency for the ideal element in this array can be obtained by inspection. The value turns out to be $\pi/4$. Thus the maximum possible efficiency of an element in this particular array is less than unity. This corresponds, of course, to the unavoidable presence of mutual coupling, as noted above.

It is now possible to derive the element realized-gain pattern for this ideal case. Since there are no grating lobes with $\lambda/2$ spacing, the $\cos\theta$ pattern shape discussed in the early sections of this paper is indeed the shape for the ideal element pattern in this case. At the peak of this pattern, the *directive* gain must be four, as mentioned in Section IV. The peak *realized* gain, however, is reduced from the directive gain by the element efficiency factor because this efficiency is the ratio of power

[8] If the phasing is π in either α or β, then two radiation lobes would just exist in the end-fire directions for this array. Since only values of θ less than 90° are considered in this paper, this is regarded as no significant formation of a grating lobe. Values of phasing greater than π are not significant because they simply correspond to negative values less than π.

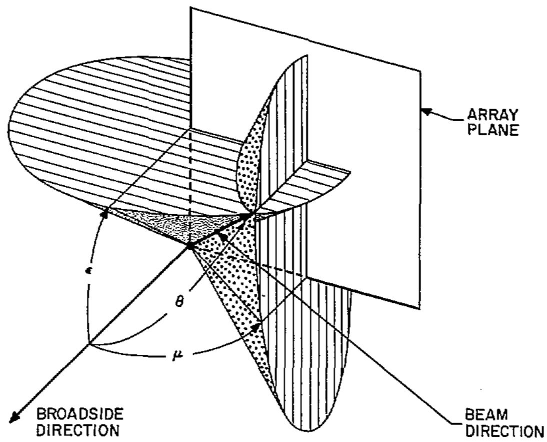


Fig. 8—Phased-array coordinate system.

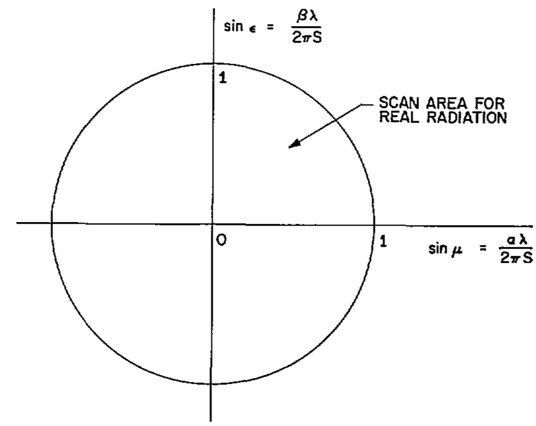


Fig. 9—Radiation circle for a square array with element spacing S.

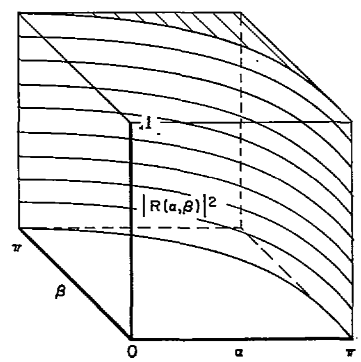


Fig. 10—The ideal characteristic of reflection vs phasing for a square array with $\lambda/2$ element spacing.

radiated to power available. Thus

$$\frac{g_r(\theta, \phi)}{g_d(\theta, \phi)} = \text{element efficiency}, \tag{15}$$

where the element realized gain g_r and the element directive gain g_d are specified in the same direction θ, ϕ. Multiplying the peak directive gain of four by the element efficiency of $\pi/4$ determined above yields a value of π for the peak realized gain of this ideal element.

To summarize the result, in an infinite square array having $\lambda/2$ element spacing, the ideal element realized-gain pattern has a peak value of π and a $\cos\theta$ shape.

B. Square Array, $\lambda/\sqrt{2}$ Spacing

When the element spacing in a square array is $\lambda/\sqrt{2}$ or more, there is no combination of phasing α, β where the array cannot radiate.[9] In this case, then, the reflection coefficient does not have to become unity at certain phasings; the ideal element can be assumed to have zero reflection coefficient for all phasings. By (11), this means that the ideal element in this particular array has *no mutual coupling* and also no passive reflection. This corresponds to the intuitive (but not always correct) specification of an ideal array element. Also, by (14), the element efficiency is unity for the ideal element in this array. Therefore, by (15), the realized gain equals the directive gain for this ideal case.

To compute the directive gain, it is necessary first to determine the shape of the element pattern. This determination is complicated by the fact that grating lobes [10] are possible with the $\lambda/\sqrt{2}$ element spacing considered here. When grating lobes can exist, the ideal element pattern no longer has the simple $\cos\theta$ shape; this is implied by the discussion in Section V and by the summation required in (8). There is actually an infinite set of pattern shapes that are possible, all of which would satisfy the ideal condition of zero reflection coefficient for all phasings. Fortunately for the brevity of this discussion, there is one simple pattern shape that is also

[9] If the phasing is π in both α and β, then four radiation lobes just exist in the end-fire directions (in the diagonal planes) for this array. For smaller values of α or β, two of these lobes can sometimes exist, and there is always at least one lobe which can radiate.

ideal in other respects; this pattern provides complete discrimination against the radiation of grating lobes.

If the main beam of the array is scanned away from the broadside direction in a cardinal plane, the grating lobe first appears in an end-fire direction on the opposite side of broadside. Further scan of the main beam will cause the grating lobe to move toward the broadside direction; eventually the grating lobe will occur at a scan angle equal to that of the main beam. If the element gain is zero for scan angles greater than this critical one, then only a single beam will radiate for any amount of phasing in the cardinal plane. The value of phasing at the critical angle is π; naturally, a bidirectional radiation pattern with symmetry about the broadside direction would be expected for this phasing.

For the more general case of scan in any plane, the critical scan angle is a function of the scan plane. However, for a square array, the critical values of phasing for either α or β are still π. Thus the scan boundary, beyond which the element gain should be zero for single-beam radiation, is a square on the $\sin \mu$, $\sin \epsilon$ plane having $-\pi$ and $+\pi$ as the values of phase in both coordinates. For element spacing of $\lambda/\sqrt{2}$, it happens that this square is inscribed in the radiation circle, as shown in Fig. 11.

If the element-gain pattern is zero for directions outside the square, the *greatest possible scanning coverage with a single lobe* is obtained because a single lobe is always radiated no matter what values of phasing are applied. Since there is radiation for all phasings, a zero reflection coefficient is possible for all phasings, even though the element pattern is truncated to preclude radiation beyond the critical scan angle. Since only one lobe is radiated, the element pattern should have the $\cos\theta$ shape for directions inside the square in order to permit a constant or zero reflection coefficient for scan angles inside the square. An attempt is made in Fig. 12 to portray this "*truncated cosine*" pattern shape which is considered here to be ideal for the infinite square array with $\lambda/\sqrt{2}$ element spacing. The truncated sections are formed by the intersection of the pattern sphere with a cluster of four 90° cones tangent to each other; these sections correspond to the four shaded regions in Fig. 11.

Having determined the pattern shape, it is a straightforward process to compute the directive gain at the peak of the pattern. This turns out to be 2π. As discussed earlier, the realized gain equals the directive gain in the ideal case, and so the peak realized gain is also 2π.

It should be mentioned that a less ideal pattern might have been chosen, that is, one which permitted grating lobes although still maintaining zero reflection coefficient for all phasings. In this case the main beam may radiate for any scan angle, and so zero reflection coefficient can also be obtained for all scan angles. The shape of this pattern would have to be modified from the simple $\cos\theta$ shape in accordance with (8); this would

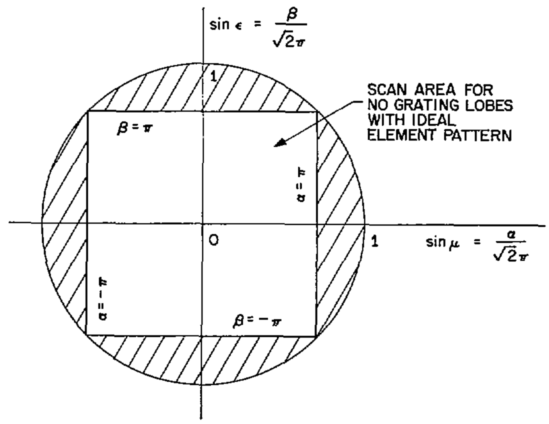


Fig. 11—Grating-lobe discrimination square for a square array with $\lambda/\sqrt{2}$ element spacing.

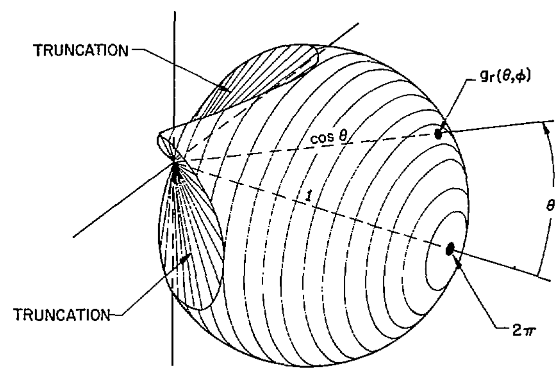


Fig. 12—The ideal element realized-gain pattern for a square array with $\lambda/\sqrt{2}$ element spacing.

involve some form of partial truncation (either stepped, or a more practical smooth shape such as in Fig. 1) in the grating-lobe regions at the sides of the pattern. The resulting peak gain would still turn out to be 2π.

To summarize the result, in an infinite square array having $\lambda/\sqrt{2}$ element spacing, the ideal element realized-gain pattern has a peak value of 2π, and a $\cos\theta$ shape truncated so as to account for, and preferably to eliminate, grating lobes.

VIII. Resolution of the Paradox

One of the results obtained in the previous section is the peak value of element realized gain for two specific cases. This was done by first determining the directive gain from the pattern shape, and then converting to realized gain. This is just the process which in Section IV, failed to yield values corresponding to those obtained from (5) based on element effective area. Now, however, it is evident that perfect agreement is obtained. This is the result of including the appropriate additional factors in the analysis. For $\lambda/2$ spacing in the square array, the additional factor is the loss by mutual coupling and passive reflection; for the $\lambda/\sqrt{2}$ spacing it is the modification of the element pattern to account for grating lobes.

These two spacings were chosen as significant examples illustrating the principles involved. It is natural to wonder whether other spacings would also yield perfect agreement; the answer is that they do. When the spacing is less than $\lambda/2$, there is more loss from mutual coupling, and the realized gain is correspondingly less. When the spacing is greater than $\lambda/\sqrt{2}$, the ideal pattern must be further truncated to prevent grating lobes, so the gain is correspondingly greater. *For intermediate spacings between* $\lambda/2$ *and* $\lambda\sqrt{2}$ *(the most common range in phased arrays), both the unavoidable mutual-coupling effects and the grating-lobe effects are present;* nevertheless, perfect agreement is still obtained.

IX. Discussion

In Section IV, it was suggested that a possible explanation of the apparent paradox regarding element gain was that the assumed perfect impedance match for all scan angles might be impossible. That paradox has been resolved without modifying the impedance-match assumption. Consequently, it appears that *perfect impedance match for all scan angles does not result in any real discrepancy in the determination of element gain.* Naturally, this suggests that it is *theoretically possible* to obtain perfect impedance match for all scan angles. Although these results have been shown for the infinite *square* array considered in this paper, they can also be shown for the currently important case of an infinite *triangular* array [16]; it therefore seems reasonable to assume that these general conclusions apply to any infinite regular array.

Achievement of such perfect match in practice is another matter. When the element spacing is large, one approach is to eliminate mutual coupling; this might be done by appropriate design of the radiating element, incorporation of isolating devices between the elements, or by cancellation networks between the transmission lines leading to the elements. When the element spacing is not large (less than $\lambda/\sqrt{2}$ for a square array) it is *not possible to eliminate mutual coupling,*[10] as was discussed in Section VII. What is needed then is just the right amounts of coupling to yield zero reflection for all scan angles; this could presumably also be accomplished by the three techniques mentioned, but would be extremely difficult.

Another approach involves the achievement of the ideal element pattern. This is perhaps the most fundamental solution, but is also likely to be the most difficult. *The most direct approach is to operate on the active impedance itself.* This may be done either externally or internally, as mentioned. An example of the *external* technique is the improved impedance match calculated [17] for a square array of dipoles, by means of baffles between the dipoles. An example of the *internal* technique is the improved match calculated [18] for a similar array, by means of circuits connecting the element lines. In the latter case it is possible to extend the method to approach perfect match for all scan angles.

In Section VII, it was shown that the ideal element pattern has a $\cos\theta$ shape for small element spacings and a truncated $\cos\theta$ shape for spacings that are not small. *The ideal element pattern for small spacings is automatically obtained by achieving impedance match for all scan angles;* for example, by using connecting circuits. However, *for large spacings, impedance match is not sufficient to provide the perfect truncation of the element pattern that would give complete discrimination against grating lobes.* Usually the larger element has a partial truncation corresponding to its inherent greater directivity, so that some discrimination against grating lobes is achieved. Often the element spacing is limited so that grating lobes are not possible for the operating range of scan angles. Nevertheless, it would be desirable to achieve the ideal element pattern with perfect truncation, thereby permitting the greatest element spacing and the greatest gain. This could be accomplished only by design of the radiating portion of the antenna, and would require either utilization of scattering by surrounding elements or superdirectivity in each element. No procedure for such a design appears to be practical.

In conclusion, the element realized-gain pattern of an infinite planar array antenna has been analyzed, including its relation to reflection vs scan angle and to grating lobes. The significance of mutual coupling and some general rules governing mutual coupling have been presented. The ideal element patterns for a square array have been derived for two element spacings. It has been shown that, for these cases, an apparent discrepancy in the gain computed from two different viewpoints is resolved when the effects of mutual coupling and grating lobes are included. The concepts presented in this paper for an *infinite* array are expected to be helpful in understanding the behavior of the radiating elements in *any regular planar array containing a large number of elements.*

[10] This law, which is exact for an infinite regular array, implies that an attempt to eliminate mutual coupling in any regular array having a large number of closely-spaced elements is unlikely to be successful. Once the net coupled power has been reduced to the minimum value obtainable with a given element spacing in an infinite array, further reduction in the noninfinite but large array with the same spacing becomes very difficult. If the mutual coupling between nearby elements in such an array is eliminated, then either the net coupling to the further-away elements or the "self coupling" (passive reflection) is likely to be increased. This writer has come across several experimental programs for the elimination of mutual coupling between closely-spaced elements in a large array, and believes that the anomalous results which were obtained are explained by this law.

X. Symbols

A = area allotted to element in array
C = equivalent backward coupling coefficient
$g(\theta, \phi)$ = gain of element in specified direction
$G(\theta, \phi)$ = gain of array in specified direction where all elements add in phase
n = number of elements (infinite)
P = power
$R(\alpha, \beta)$ = equivalent reflection coefficient when all elements are excited in a specified phase
S = element spacing
α = excitation phase between adjacent elements in row
β = excitation phase between adjacent elements in column
θ = angle from broadside direction (scan angle)
ϕ = angle of plane from row (scan-plane angle)
μ = angle of cone from column
ϵ = angle of cone from row
λ = wavelength
sub d = directive
sub m = lobe number
sub $_0$ = single-lobe
sub p = column number of element
sub q = row number of element
sub r = realized.

XI. Acknowledgment

The writer would like to acknowledge the cooperation of Bell Telephone Laboratories in the preparation of this paper, and especially to thank P. H. Smith. At Wheeler Laboratories, helpful comments and suggestions have been received from H. A. Wheeler, N. A. Spencer, H. W. Redlien, D. S. Lerner, and G. H. Knittel.

References

[1] J. L. Allen, private communication; May, 1963. See also J. L. Allen, *et al.*, "Phased Array Radar Studies," M.I.T. Lincoln Lab., Lexington, Mass., Tech. Rept. No. 299, pp. 159–168; February, 1963.
[2] H. A. Wheeler, "The radiation resistance of an antenna in an infinite array or waveguide," Proc. IRE, vol. 36, pp. 478–487; April, 1948.
[3] J. L. Allen, "Gain and impedance variation in scanned dipole arrays," IRE Trans. on Antennas and Propagation, vol. AP-10, pp. 566–572; September, 1962.
[4] R. E. Willey, "Space tapering of linear and planar arrays," IRE Trans. on Antennas and Propagation, vol. AP-10, pp. 369–377; July, 1962.
[5] IRE Standards on Antennas, Definitions of Terms; 1948.
[6] L. I. Parad, "Some mutual impedance effects in phased arrays," *Microwave J.*, vol. V, pp. 87–89; June, 1962.
[7] L. A. Kurtz and R. S. Elliott, "Systematic errors caused by the scanning of antenna arrays: phase shifters in the branch lines," IRE Trans. on Antennas and Propagation, vol. AP-4, pp. 619–627; October, 1956.
[8] J. Blass and S. J. Rabinowitz, "Mutual coupling in two-dimensional arrays," 1957 IRE WESCON Convention Record, pt. I, pp. 134–150.
[9] J. D. Kraus, "Antennas," McGraw-Hill Book Co., Inc., New York, N. Y., pp. 17–18; 1950.
[10] W. H. Von Aulock, "Properties of phased arrays," Proc. IRE, vol. 48, pp. 1715–1727; October, 1960.
[11] P. S. Carter, Jr., "Mutual impedance effects in large beam scanning arrays," IRE Trans. on Antennas and Propagation, vol. AP-8, pp. 276–285; May, 1960.
[12] S. Edelberg and A. A. Oliner, "Mutual coupling effects in large antenna arrays: Part I—Slot arrays," IRE Trans. on Antennas and Propagation, vol. AP-8, pp. 286–297; May, 1960.
[13] P. W. Hannan, P. J. Meier, and M. A. Balfour, "Simulation of phased array antenna impedance in waveguide," IEEE Trans. on Antennas and Propagation (*Correspondence*), vol. AP-11, pp. 715–716; November, 1963.
[14] R. V. Churchill, "Fourier Series and Boundary Value Problems," McGraw-Hill Book Co., Inc., New York, N. Y., pp. 116–118; 1941.
[15] F. C. Ogg, Jr., "Steerable array radars," IRE Trans. on Military Electronics, vol. MIL-5, pp. 80–94; April, 1961.
[16] E. D. Sharp, "A triangular arrangement of planar-array elements that reduces the number needed," IRE Trans. on Antennas and Propagation, vol. AP-9, pp. 126–129; March, 1961.
[17] S. Edelberg and A. A. Oliner, "Mutual coupling effects in large antenna arrays II: Compensation effects," IRE Trans. on Antennas and Propagation, vol. AP-8, pp. 360–367; July, 1960.
[18] P. W. Hannan, D. S. Lerner, and G. H. Knittel, "Wide-angle impedance matching calculated for a phased array antenna," *Digest of PTGAP Internat'l Symp.*, pp. 228–233; July, 1963.