Singularity Evaluation of the Straight-Wire Mixed-Potential Integral Equation in the Method of Moments Procedure

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***Abstract*—A rigorous treatment on the computation of the various integrals that arise in the method of moments (MoM) formulation of the straight-wire electric field integral equation is provided. For triangle basis functions along with delta function, pulse or triangle weights, particular attention is given to integrals whose integrands are weakly singular. A singularity extraction technique is employed that splits the integral under question into two parts: one that is numerically integrable and one that is analytically integrable. Closed-form approximations based on Taylor series techniques are also provided for the former. These approximations are very robust resulting in errors less than 0.1% when $2a/\delta < 1$; here δ is the length of a single MoM segment and a is the wire radius. Results are compared with data from the literature to demonstrate the robustness of the presented approach for fat wires.**

***Index Terms*—Basis function, impedance, kernel, singularity.**

I. Introduction

The determination of the current distribution of a straight wire antenna has a rich history both in terms of the variational and method of moments formulations. In either case, the solutions for dealing with the problematic singularity that arise in these formulations have been numerous. For example, Schelkunoff suggested splitting the integrand of the cylindrical kernel into a continuous frequency-dependant term and a singular non-frequency dependant component [1]. Harrington gave a series formulation for the cylindrical kernel based on a Maclaurin series [2], and provided a thin-wire approximation based on the reduced kernel approach which results in a singularity-free formulation. Butler [3] addressed the singularity in the cylindrical kernel and obtained a thin-wire result for the potential (integral of the kernel) in cases where $2a/\delta < 1$, where a is the wire radius, and δ is the size of a single segment in the method of moments procedure. W. Wang found an exact formulation of the kernel using spherical Hankel functions [4], and D.H. Werner used the result from Wang to obtain an exact formulation of the vector potential and electric field of a cylindrical antenna with uniformly distributed current and arbitrary radius [5]. Furthermore, Werner computed integrals of the kernel using a series formulation involving exponential integrals and higher-order associated integrals. Werner, Hoffman, and Werner [6] provided an improvement of the thin-wire kernel approximation with the restriction that $a/\delta < 0.25$. Heldring and Rius used a transformation of variables to obtain a solution for the full-kernel utilizing the elliptic integral of the first kind [7]. Using a slightly different approach, Mohan and Weile formulated the problem in terms of zero and first order moments of the kernel [8], and generalized the technique for higher order moments [9], the results of which are suitable for Gauss-quadrature numerical integration. Using sinusoidal basis functions and delta weights, D. H. Werner [10] provided a formulation for moderately thick cylindrical wire antennas $0.01\lambda \leq a \leq 0.1\lambda$.

The work described herein falls under the class of kernel methods associated with singularity extraction. Restricting our attention to triangle basis functions and pulse weights, we are able to reformulate the method of moment equations in a way that leads to a robust computational procedure without making any simplifying assumptions, particularly with respect to wire radius or length. Like most formulations in the literature, we exclude end cap effects. However, experimental data by R. W. P. King [11] suggests that end cap effects are negligible even for fat wires. Closed-form approximations are also given for the self-impedance terms that are accurate to less than 0.1% for certain cell sizes. Our approach is in contrast to that of Wilton and Champagne [12] who reformulated the equations using singularity cancellation to deal with the non-analyticity of the distance function R raised to odd powers. Such an approach is essential if the objective is to devise a quadrature numerical integration scheme that efficiently converges with increasing polynomial order, as in the case with higher-order modeling. Since the approach described herein considers only low-order basis functions and weights, high-order quadrature efficiency is unnecessary. As shown in this paper, for a sufficient number of basis functions, high accuracy is maintained using singularity extraction and the resulting scheme is straightforward to implement. Input impedance results are provided for the fat wire case and compared with published data [10] to verify the claims made herein.

II. Wire Antenna

A. Straight Wire Integral Equation

Consider a straight, two-wire antenna of radius a and length $2l$ that is coaxial with the z-axis of some coordinate system. An

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impressed electric field $E^i(z)$ in the gap between the wires induces an axial current $I(z)$ on the wires such that the total electric field on the surface of the wire is zero. To find this induced current in terms of the gap field, we must solve the following well-known integral equation:

$$4\pi j\omega\varepsilon E^i(z) = -k^2 \int_{-l}^{l} I(z')K(z,z')dz' - \frac{d}{dz}\int_{-l}^{l} \frac{dI(z')}{dz'}K(z,z')dz'. \quad (1)$$

Here $K(z,z')$ is the kernel of the integral equation, which is given by

$$K(z,z') = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \frac{e^{-jkR}}{R}d\phi' \quad (2)$$

where $k = 2\pi/\lambda$ is the wavenumber and

$$R = R(z,z',\phi') = \sqrt{(z-z')^2 + 4a^2\sin^2(\phi'/2)}. \quad (3)$$

It is customary to represent the induced current in terms of a linear combination of basis functions $B_n(z)$, i.e.

$$I(z') = \sum_{n=1}^{N} I_n B_n(z'). \quad (4)$$

The coefficients I_n are then determined by the method of weighted residuals in conjunction with the weighting functions $w_m(z)$. The outcome of this process is a matrix equation of the form $\mathbf{V} = \mathbf{ZI}$, where

$$\begin{aligned} \mathbf{I} &= [I_1 \quad I_2 \quad I_3 \quad \dots \quad I_N]^t \\ \mathbf{V} &= [V_1 \quad V_2 \quad V_3 \quad \dots \quad V_N]^t \end{aligned} \quad (5)$$

and

$$\mathbf{Z} = \begin{pmatrix} Z_{11} & \dots & Z_{1N} \\ \vdots & \ddots & \vdots \\ Z_{N1} & \dots & Z_{NN} \end{pmatrix}. \quad (6)$$

By definition

$$V_m = \int_{-l}^{l} w_m(z)E^i(z)dz \quad (7)$$

and

$$Z_{mn} = \frac{-k^2}{4\pi j\omega\varepsilon}\int_{-l}^{l} w_m(z)\int_{-l}^{l} B_n(z')K(z,z')dz'dz - \frac{1}{4\pi j\omega\varepsilon}\int_{-l}^{l} w_m(z)\frac{d}{dz}\int_{-l}^{l}\frac{dB_n(z')}{dz'}K(z,z')dz'dz. \quad (8)$$

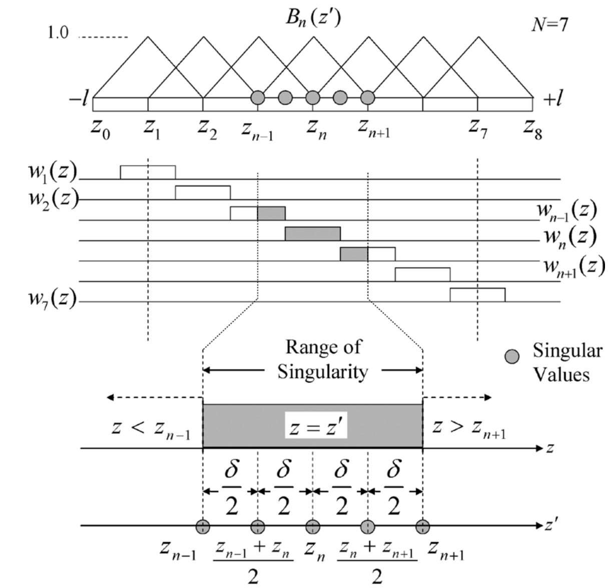

Fig. 1. A depiction of triangle basis functions, pulse weights, and the corresponding singularity range in Z_{mn}.

For purposes of this paper, we employ triangle basis functions and pulse weights

$$B_n(z') = \begin{cases} (z'-z_{n-1})/\delta; & z_{n-1} \le z' \le z_n \\ (z_{n+1}-z')/\delta; & z_n \le z' \le z_{n+1} \\ 0; & \text{otherwise} \end{cases} \quad (9)$$

and

$$w_m(z) = \begin{cases} 1; & z_m - \delta/2 \le z \le z_m + \delta/2 \\ 0; & \text{otherwise} \end{cases}. \quad (10)$$

In writing the above equations, we have assumed that the weights span one cell of size $\delta = z_n - z_{n-1}$, whereas the basis functions span two cells. The situation is depicted in Fig. 1. Upon the insertion of these functions into (8), we find that

$$Z_{mn} = \frac{-k^2}{4\pi j\omega\varepsilon\delta}\int_{z_m-\delta/2}^{z_m+\delta/2} \Lambda_n(z)dz - \frac{1}{4\pi j\omega\varepsilon\delta}\Gamma_n(z_m) \quad (11)$$

where

$$\Lambda_n(z) = Q_n(z) - z_{n-1}P_n(z) + z_{n+1}P_{n+1}(z) - Q_{n+1}(z) \quad (12)$$

and

$$\Gamma_n(z) = P_n\left(z+\frac{\delta}{2}\right) - P_n\left(z-\frac{\delta}{2}\right) - P_{n+1}\left(z+\frac{\delta}{2}\right) + P_{n+1}\left(z-\frac{\delta}{2}\right). \quad (13)$$

It is observed in the above equation that the evaluation of Z_{mn} rests exclusively on the evaluation of the two integrals $P_n(z)$ and $Q_n(z)$, where

$$P_n(z) = \int_{z_{n-1}}^{z_n} K(z,z')dz' \tag{14}$$

and

$$Q_n(z) = \int_{z_{n-1}}^{z_n} z'K(z,z')dz'. \tag{15}$$

The contribution of this paper focuses on the proper numerical evaluation of these two integrals.

Since the integrand of $K(z,z')$ is singular when $R=0$, it is common to decompose the integrand of $P_n(z)$ in terms of a time-harmonic integrand and a static integrand, with the latter embedding the singularity

$$P_n(z) = S_n(z) + J_n(z) \tag{16}$$

where

$$J_n(z) = \frac{1}{2\pi}\int_{z_{n-1}}^{z_n}\int_{-\pi}^{\pi} \frac{e^{-jkR}-1}{R} d\phi' dz' \tag{17}$$

and

$$S_n(z) = \frac{1}{2\pi}\int_{z_{n-1}}^{z_n}\int_{-\pi}^{\pi} \frac{d\phi' dz'}{R}. \tag{18}$$

Given that the integrand of $J_n(z)$ is non-singular, its evaluation by numerical methods is straightforward, albeit with loss of efficiency as quadrature order is increased, as noted in the Introduction.

As for $Q_n(z)$, it behooves us to deal with the singularity at $R=0$ by separating this integral into three parts through the process of adding and subtracting a z'/R term, and by adding and subtracting a z/R term. That is,

$$\frac{z'e^{-jkR}}{R} = z'\left(\frac{e^{-jkR}-1}{R}\right) + \frac{z'-z}{R} + \frac{z}{R} \tag{19}$$

in which case

$$Q_n(z) = G_n(z) + F_n(z) + zS_n(z) \tag{20}$$

where $S_n(z)$ was previously given by (18). Also,

$$G_n(z) = \frac{1}{2\pi}\int_{z_{n-1}}^{z_n} z' \int_{-\pi}^{+\pi} \frac{e^{-jkR}-1}{R} d\phi' dz' \tag{21}$$

and

$$F_n(z) = \frac{1}{2\pi}\int_{z_{n-1}}^{z_n}\int_{-\pi}^{\pi} \frac{z'-z}{R} d\phi' dz'. \tag{22}$$

We see that the integrand of $G_n(z)$ is bounded over the entire range of integration; for this reason, we may employ numerical integration with ease. As for $F_n(z)$, its integrand is weakly singular, but the integration through the singularity can be accomplished in closed-form. It should be noted that the reduced kernel formulation is easily obtained by setting $4\sin^2(\phi'/2)$ to unity in (3), resulting in a kernel of the form e^{-jkR}/R, with $R = \sqrt{(z-z')^2 + a^2}$. When this is done none of the aforementioned integrals have singular integrands, in which case all integrals can be evaluated somewhat straightforwardly by numerical integration.

B. Computation of $S_n(z)$ and $F_n(z)$

In this section special attention is given to the evaluation of the two problematic integrals $S_n(z)$ and $F_n(z)$. Consider the former, as given by (18). Interchanging the order of integration and making use of the identity

$$\int \frac{dx}{\sqrt{x^2+b^2}} = \ln(x + \sqrt{x^2+b^2}) \tag{23}$$

we find that

$$S_n(z) = T(z - z_{n-1}) - T(z - z_n) \tag{24}$$

where

$$T(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\left[x + \sqrt{x^2 + t^2(\phi')}\right] d\phi' \tag{25}$$

with $t(\phi') = 2a\sin(\phi'/2)$. We observe that the above integrand is weakly singular when $x<0$ and when $t=0$. For the case of triangle basis functions and pulse weights in the method of moments procedure, the singularity is encountered when $z_{n-1} \le z \le z_{n+1}$, as depicted in Fig. 1.

First consider the situation when $x=0$, in which case from (25)

$$T(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln|2a\sin(\phi'/2)|\, d\phi'. \tag{26}$$

Equally so, by virtue of symmetry in ϕ'

$$T(0) = \ln(2a) + \frac{2}{\pi}\int_{0}^{\pi/2} \ln(\sin\phi')d\phi'. \tag{27}$$

A closed—form solution for the integral of (27) is given by Gradshteyn and Ryzhik #4.224.3 [13], in which case

$$T(0) = \ln(2a) - \ln(2) = \ln(a). \tag{28}$$

Next consider the situation when $x<0$. For this case, (25) is equivalent to

$$T(x)|_{x<0} = \ln|x| + \frac{2}{\pi}\int_{0}^{\pi/2} \ln\left[-1 + \sqrt{1+\alpha^2(x)\sin^2\phi'}\right] d\phi' \tag{29}$$

where

$$\alpha(x) = 2a/x. \tag{30}$$

Clearly, the integrand in (29) is weakly singular when $\phi' = 0$. But this singularity can be removed by recognizing that if $|\alpha \sin\phi'| \ll 1$, then

$$\sqrt{1+\alpha^2\sin^2\phi'} \approx 1 + \frac{1}{2}\alpha^2\sin^2\phi' \tag{31}$$

in which case

$$\ln[-1+\sqrt{1+\alpha^2\sin^2\phi'}] \approx \ln\left(\frac{1}{2}\alpha^2\sin^2\phi'\right). \tag{32}$$

By adding and subtracting the integral of this limiting term to and from $T(x)$, we can reformulate $T(x)$, without approximation, as follows:

$$T(x)|_{x<0} = \ln|x| + T' + \frac{2}{\pi}\int_0^{\pi/2} \ln\left[\frac{-1+\sqrt{1+\alpha^2\sin^2\phi'}}{(\alpha^2/2)\sin^2\phi'}\right] d\phi' \tag{33}$$

where

$$T' = \frac{2}{\pi}\int_0^{\pi/2} \ln\left[(\alpha^2/2)\sin^2(\phi')\right] d\phi'. \tag{34}$$

Here we note from either a Taylor analysis or from L'Hopital's rule that the integrand in (33) is non-singular, even when $\phi' = 0$. The integrand of T' is singular at $\phi' = 0$, but integrable in closed-form. Specifically,

$$T' = \frac{4}{\pi}\int_0^{\pi/2} \ln(\sin\phi')d\phi' + \frac{2}{\pi}\int_0^{\pi/2} \ln(\alpha^2/2)d\phi'. \tag{35}$$

The integration of the second integral is trivial; the first integral is accomplished again via Gradshteyn and Ryzhik #4.224.3. Hence,

$$T' = \ln(\alpha^2/2) - 2\ln(2) = \ln(\alpha^2/8). \tag{36}$$

This equation inserted into (33) yields

$$T(x)|_{x<0} = \ln|x| + \ln(\alpha^2/8) + \frac{2}{\pi}\int_0^{\pi/2} \ln\left[\frac{-1+\sqrt{1+\alpha^2\sin^2\phi'}}{(\alpha^2/2)\sin^2\phi'}\right] d\phi'. \tag{37}$$

Given that the above integrand is non-singular and smoothly varying, numerical integration is straightforward to implement.

The expression for $T(x)$ in (37) is exact for any wire radius. Any approximations in its value will be associated with the numerical implementation of the integration. However, a very good closed-form approximation can be given for the thin-wire case, which is not an uncommon condition for wire antenna calculations in the method of moments procedure. This approximation is achieved using the Taylor series method.

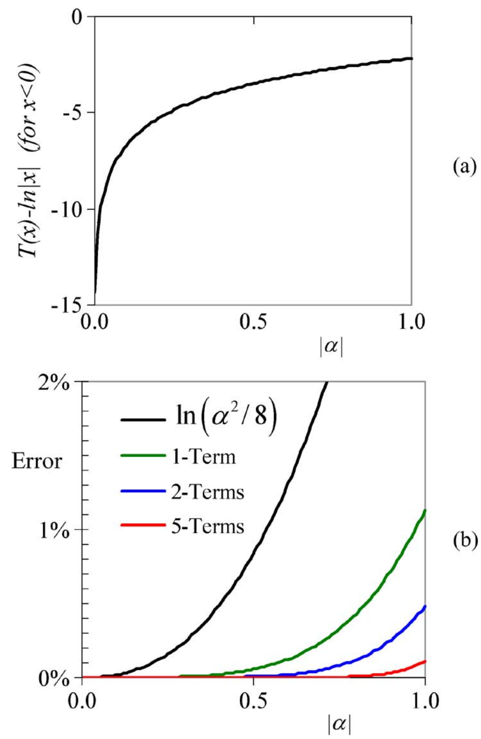

Fig. 2. (a) Plot of (37) versus α using the direct numerical integration method and (b) the percent error between (37) and (39) when numerical integration is used to evaluate (37).

Substituting the expansion for the term $\sqrt{1+\alpha^2\sin^2\phi'}$ into the expression for $T(x)|_{x<0} - \ln|x|$, factoring out $(\alpha^2\sin^2\phi')/2$, followed by the Taylor expansion of $\ln(1+u)$, and use of the multinomial theorem yields a generalized series expression. However, retention of the first five terms is typically enough to generate an accurate answer, in which case

$$\ln\left(\frac{-1+\sqrt{1+u^2}}{u^2/2}\right) = A_1u^2 + A_2u^4 + A_3u^6 + A_4u^8 + A_5u^{10} + O(u^{12}) \tag{38}$$

where $u = \alpha\sin\phi'$, $A_1 = -1/4$, $A_2 = 3/32$, $A_3 = -5/96$, $A_4 = 35/1024$, and $A_5 = -63/2560$. Upon integrating (38), and using #3.621-3 in Gradshteyn and Ryzhik [13], we obtain

$$T(x)|_{x<0} = \ln|x| + \ln\left(\frac{\alpha^2}{8}\right) + \sum_{n=1}^{5} A_n\frac{(2n)!\alpha^{2n}}{(2^n n!)^2} + \mathrm{O}(\alpha^{12}). \tag{39}$$

Numerical calculations of (37), and (39) are shown in Fig. 2. Including only the first term in the series of (39) (which varies as α^2) and ignoring α^4—terms and higher, we see that the error in computing $T(x) - \ln|x|$ is less than 1.28% at $\alpha = 1$. The error decreases to 0.1% (also at $\alpha = 1$) with 5-terms in the series, i.e., ignoring the α^{12} term and higher. In fact, the error is only 5% at $\alpha = 1$ when retaining the $\ln(\alpha^2/8)$ term only.

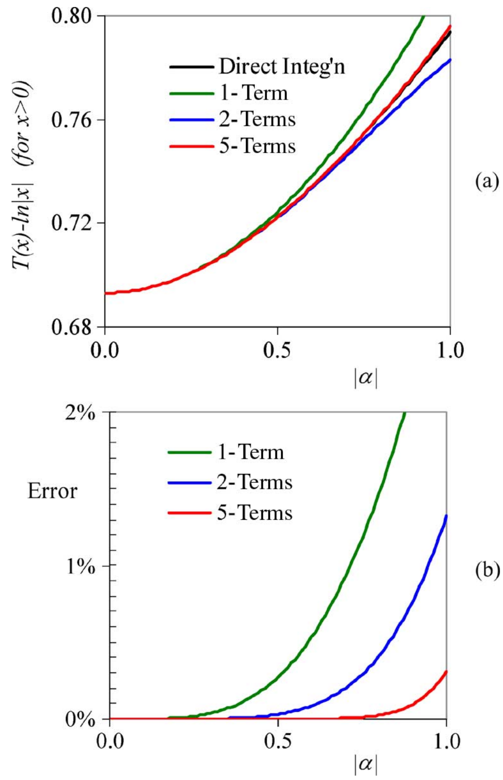

Fig. 3. (a) Plot of (40) versus α using the direct numerical integration method and (b) the percent error between (40) and (42) when numerical integration is used to evaluate (40).

Now let us turn our attention to $T(x)$ when $x > 0$. Although the integrand of (25) is technically non-singular, it is nevertheless nearly singular when $x \approx 0$ and $\phi' = 0$. For this case, we replace (25) with

$$T(x)|_{x>0} = \ln x + \frac{2}{\pi} \int_0^{\pi/2} \ln[1 + \sqrt{1 + \alpha^2 \sin^2 \phi'}] d\phi'. \quad (40)$$

Expanding the integrand in (40) in a Taylor series approximation using Gradshteyn and Ryzhik #1.515.1 [13] and assuming that $\alpha^2 \sin^2 \phi' \leq 1$, we find that

$$\ln(1 + \sqrt{1 + \alpha^2 \sin^2 \phi'}) = \ln(2) - \sum_{m=1}^{\infty} (-1)^m \frac{(2m-1)!}{2^{2m}(m!)^2} (\alpha^2 \sin^2 \phi')^m. \quad (41)$$

Integrating (41) and using #3.621-3 in Gradshteyn and Ryzhik [13], we obtain

$$T(x)|_{x>0} = \ln x + \ln 2 - \sum_{m=1}^{\infty} (-1)^m \frac{(2m-1)!(2m)!}{(2^m m!)^4} \alpha^{2m}. \quad (42)$$

Fig. 3 shows a plot of $T(x)|_{x>0} - \ln x$ computed by direct integration, of (40), and using the series approach in (42).

As expected, the error between the series and direct integration approach decreases as more terms are included in the series. More specifically, when $T(x)|_{x>0} - \ln x$ is evaluated at $\alpha = 1$, the error is 3.1% with one term in the series, less than 1.3% with 2 terms, and less than 0.03% with 5 terms. A similar result to (42), sans numerical quantification, was reported by Butler [3].

In summary, $S_n(z)$ is defined by (18) and reformulated using (24), with the latter being a function of (28), (37) and (40). Approximations for (37) and (40) are given by (39) and (42), respectively. When the integrand is singular within the integration range, (24) must be used; otherwise, one can use either (18) or (24). However, due to the non-analyticity of the square root and logarithmic functions in (37), (18) is more robust for non-singular integrations, particularly when α is large. Furthermore, a thin-wire approximation is obtained for small values of α, in which $S_n(z)$ may be approximated by

$$S_n(z) \approx \begin{cases} \ln(2\delta/a); & z = z_{n-1} \text{ or } z_n \\ 2\ln(\delta/a); & z = (z_{n-1} + z_n)/2 \end{cases}. \quad (43)$$

We may use direct numerical integration of (18) when z is outside the interval $[z_{n-1}, z_n]$.

Finally, the remaining integral of interest is

$$F_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{z_{n-1}}^{z_n} \frac{z' - z}{\sqrt{(z'-z)^2 + t^2(\phi')}} dz' d\phi'. \quad (44)$$

The integrand of $F_n(z)$ is weakly singular, but the integration through the singularity can be accomplished in closed-form; the result is

$$F_n(z) = U(z - z_n) - U(z - z_{n-1}) \quad (45)$$

where

$$U(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sqrt{x^2 + 4a^2 \sin^2(\phi'/2)} d\phi' = \frac{\sqrt{x^2 + 4a^2}}{\pi} E\left(\pi, \frac{2a}{\sqrt{x^2 + 4a^2}}\right). \quad (46)$$

Here $E(\varphi, \beta)$ is the elliptic integral of the second kind, per Gradshteyn and Ryzhik #8.111-3 [13]. Furthermore,

$$U(0) = \frac{2a}{2\pi} \int_{-\pi}^{\pi} |\sin(\phi'/2)| \, d\phi' = \frac{4a}{\pi}. \quad (47)$$

Two additional MoM formulations are worth mentioning: the delta-weight, in which Z_{mn} is given by

$$Z_{mn} = \frac{j30}{k\delta} \left[k^2 \Lambda_n(z_m) + \frac{\Gamma_n(z_m)}{\delta} \right] \quad (48)$$

and the Galerkin method in which the weight is equal to the basis function. For this latter case

$$Z_{mn} = \frac{j30k}{\delta^2} \int_{z_{m-1}}^{z_m} (z - z_{m-1}) \Lambda_n(z) dz + \frac{j30k}{\delta^2} \int_{z_m}^{z_{m+1}} (z_{m+1} - z) \Lambda_n(z) dz$$

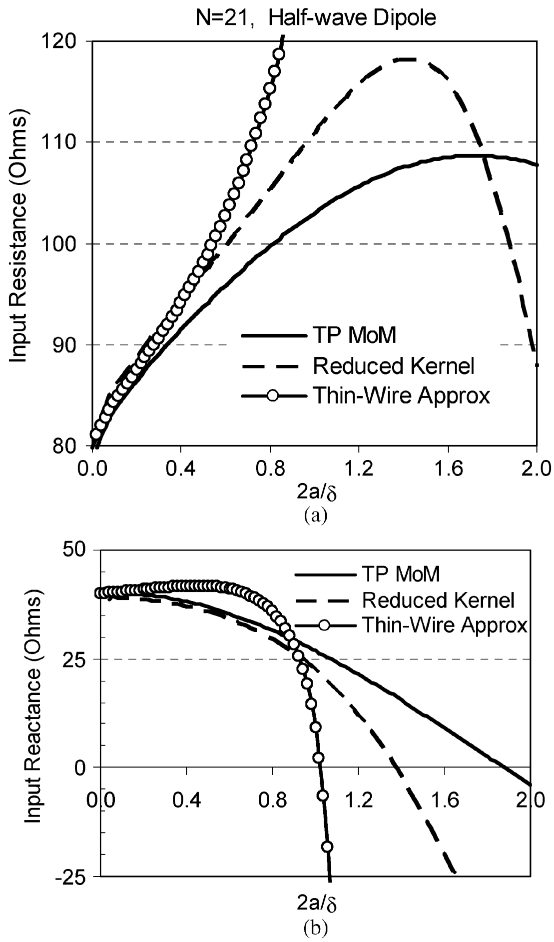

Fig. 4. A comparison of the input resistance (a) and the input reactance (b) of a half-wave dipole computed using triangle basis and pulse weights (TP MoM). The data for the thin-wire version of the TP MoM method and the reduced kernel method are also shown for comparison.

$$+\frac{j30}{k\delta^3}\int_{z_{m-1}}^{z_m}(z-z_{m-1})\Gamma_n(z)dz$$
$$+\frac{j30}{k\delta^3}\int_{z_m}^{z_{m+1}}(z_{m+1}-z)\Gamma_n(z)dz. \tag{49}$$

We note that no new integrals are defined in (48) and (49), and the integrations can be easily computed using Simpson's or Gauss quadrature rules.

III. Numerical Results

We translated the key equations from this paper into a MoM code using both Simpson's and Gauss quadrature rules to perform the numerical integrations. The results obtained confirm the validity of our approach. The first confirmation was the replication of input impedance plots (both real and imaginary) versus electrical length of a dipole for a given thickness, and the results were in agreement with published data [14]. These results are not shown here as they are well-known.

We then compared the input impedance of a half-wave dipole using the reduced kernel and the thin-wire approaches, with the triangle basis and pulse weight MoM method (denoted by the abbreviation TP MoM).

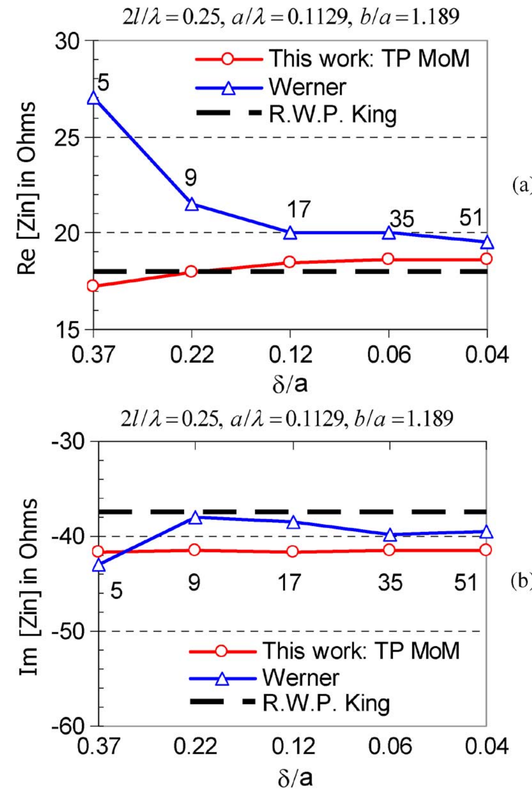

Fig. 5. Input resistance (a) and input reactance (b) for a moderately-thick quarter-wave dipole at 300 MHz using a fixed gap source model.

The plots in Fig. 4(a) and (b) were generated using a delta gap source. This results in a voltage column vector whose elements are zero, except for the center element whose value is equal to the gap voltage (1 volt in our case). Fig. 4 shows that all three techniques converge to the same answer for thin-wires, and diverge from one another for thicker wires. In fact, the reduced kernel method does better than the thin-wire approach of (43) in predicting the dipole input impedance.

We also tested the validity of this approach (triangle basis with pulse weights) for moderately thick wires. This case was tested by analyzing the same dipoles that were considered by Werner [10] and by using a fixed-gap source [15]. The first example was a quarter-wave dipole having a wire radius $a = 0.1129\lambda$ and a gap width $b/a = 1.189$. The second example was a half-wave dipole of the same gap width and a wire radius $a = 0.0509\lambda$. The input impedance was computed versus the ratio δ/a using different numbers of basis functions (marked on each plot). The plotted data shows results obtained from this work, Werner's result, and R.W.P. King's published data [11]. Fig. 5(a) and (b) show the results for the quarter-wave dipole, and Fig.6(a) and (b) show the results obtained for the half-wave dipole case.

The data in Figs. 5 and 6clearly show that the proposed method in this paper produces good results in comparison with published data by D. H. Werner's and R.W.P. King. In particular, for the half-wave dipole case, shown in Fig. 6, excellent convergence was achieved by using only 5 basis functions for

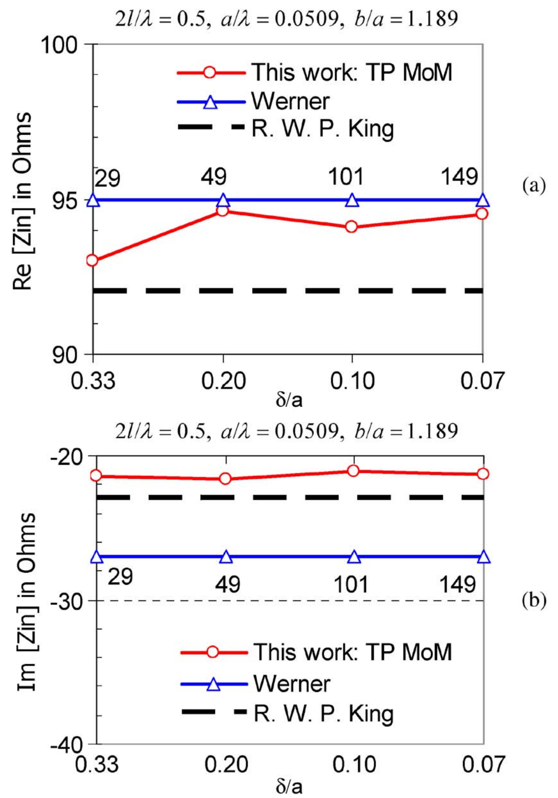

Fig. 6. Input resistance (a) and input reactance (b) for a moderately-thick half-wave dipole at 300 MHz using a fixed gap source model.

the real part and about 9 basis functions for the imaginary part. In other words, we achieved good accuracy with a low number of basis functions. Furthermore, the attractiveness of the method is in the simplicity of numerical implementation, where we only need to evaluate a few integrals with smoothly varying integrands. We surmise that the poor convergence of [10] as seen in Fig. 5(a), is due to the choice of delta function weights. Although such weights result in a certain amount of simplicity, the combined order of weights and basis functions is less than the combined order of the method herein. Per [16] the combined order of the basis and weight functions was shown to have a strong effect on the overall accuracy. Also, it is not clear from the work of King how accurate the impedance measurements are. The true answer may be in fact closer to the data provided by us and by [10].

IV. Conclusions

The method of moments solution of the straight-wire dipole integral equation problem requires the evaluation of four integrals, two of which have integrands that are weakly singular. This paper has provided a rigorous and accurate procedure for dealing with these singular integrands regardless of antenna length or radius. However, when the parameter $2a/\delta$ is less than unity, which typically occurs near the main diagonal of the MoM matrix for thin wire geometries, approximations for these integrals have also been given that are accurate to 0.1% or less. When used in the method of moment procedure, excellent results for the current distribution and input impedance are obtained.

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