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On the Integral Equations of Thin Wire Antennas

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***Abstract*—The feasibility of direct numerical calculations of antenna integral equations is investigated. It is shown that integral equation of Hallen's type is the most adequate for such applications. The extension of Hallen's integral equation to describe thin wire antennas of arbitrary geometry is accomplished, and results are presented for dipole, circular loops, and equiangular spiral antennas.**

Introduction

During the past seven years, the advancement of antenna design has been characterized by an exhaustive utilization of antenna geometry. Broadband antennas are notable examples. In the study of antenna theory, a knowledge of the current distribution is of fundamental importance. Such data may be obtained either by measurement or by solving the antenna integral equation. Integral equations are difficult to solve even for the simplest case of a dipole antenna. However, as a result of the development in modern high speed computers, the range of application of the integral equation method has been greatly enlarged. The purpose of this paper is to present an investigation of the feasibility of direct numerical calculations of antenna integral equations. To simplify the discussion, the trapezoidal rule of integration is assumed throughout, although it is realized that in a practical calculation better integration schmes, such as quadratic rule, etc., may need to be used. Typical results of calculations are presented.

Numerical Solutions of Dipole Antennas

It is well known that the axial component of the electric field produced by the current on a cylindrical dipole antenna [1] is given by

$$\frac{d}{dz}\int_L \oint \frac{dJ(z')}{dz'} G(z, c; z'c')dc'dz' + k^2 \int_L \oint J(z')G(z, c; z', c')dc'dz' = j\omega\epsilon E(z) \quad (1)$$

where the symbol $J(z')$ represents the surface current density, $\oint dc'$ represents the integration around the periphery of the cylinder, and $G(z, c; z', c')$ is the free space Green's function,

$$G(z, c; z', c') = \frac{e^{-jk|\bar{r}-\bar{r}'|}}{4\pi|\bar{r} - \bar{r}'|}$$

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For simplicity we shall omit the integration $\oint dc'$ in the discussion that follows, i.e., the symbol $\int_L dz'$ will represent the surface integral over the cylinder.

When the electric field on the surface of the antenna is considered, (1) reduces to

$$\frac{d}{dz}\int_L J'(z')G(z, z')dz' + k^2 \int_L J(z')G(z, z')dz' = -j\omega\epsilon E_z^i(z) \quad (2)$$

where $E_z^i(z)$ is the electric field produced by the generator. Equation (2) is an integrodifferential equation for the current, which may be solved numerically by a combination of the difference equation method and the numerical integration method. The disadvantage of such an approach is that difference equations are generally unstable and critical to the errors in the approximation. An alternative approach is to transform (2) into a pure integral equation. Equation (2) may be readily transformed into such an equation of one of several familiar forms. The one used by Pocklington [2] is

$$\int_L J(z')\left[\frac{\partial^2}{dz^2} G(z, z') + k^2 G(z, z')\right] dz' = -j\omega\epsilon E_z^i(z) \quad (3)$$

integrating both sides of (3), say from 0 to z, gives

$$\int_L J(z')\left[\frac{\partial}{\partial z} G(z, z') + k^2 \int_0^z G(\xi, z')d\xi\right] dz' = -j\omega\epsilon \int_0^z E_z^i(\xi)d\xi + A \quad (4)$$

The integral equation used by Hallen [1] is,

$$\int_L J(z')G(z, z')dz' = B\cos kz - \frac{jV}{2Z_0}\sin k|z| \quad (5)$$

In these integral equations, the constants of integration A and B are to be determined by the condition that the current vanishes at both ends of the antenna; V and Z_0 are, respectively, the voltage applied and the intrinsic impedance of free space.

The numerical solution of an integral equation may be effected by approximating the integration with a finite sum at n different points. The resulting algebraic equations will have the following form [3], [4]:

$$\begin{aligned}
K_{11}J(z_1) + K_{12}J(z_2) + \cdots + K_{1n}J(z_n) &= F(z_1)\\
K_{21}J(z_1) + K_{22}J(z_2) + \cdots + K_{2n}J(z_n) &= F(z_2)\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots&\cdots\\
K_{n1}J(z_1) + K_{n2}J(z_2) + \cdots + K_{nn}J(z_n) &= F(z_n) \quad (6)
\end{aligned}$$

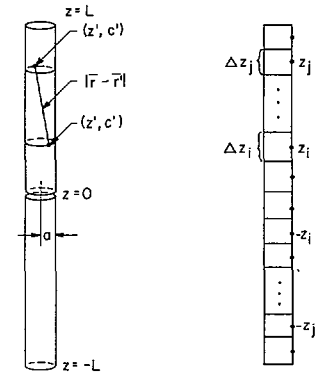

Fig. 1. Relevant of a dipole antenna and its subdivisions

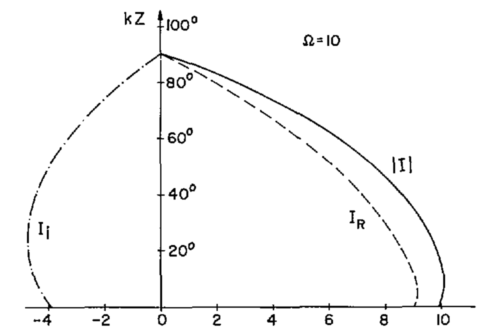

Fig. 2. Current distribution $I=I_R+jI_i$ on a dipole antenna of parameters $\Omega=2\log 2L/a=10$, $kL=\pi/2$

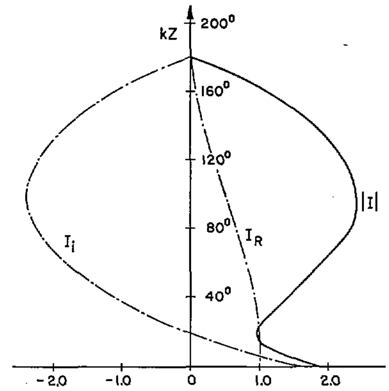

Fig 3. Current distribution on a dipole antenna of parameters $\Omega=2\log 2L/a=10$, $kL=\pi$.

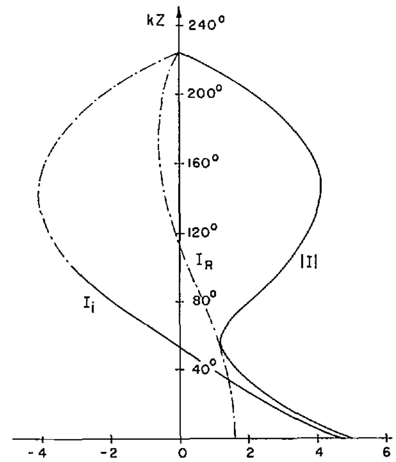

Fig. 4. Current distribution $I=I_R+jI_i$ on a dipole antenna of parameters $\Omega=2\log 2L/\theta=10$, $kL=5\pi/4$.

The matrix elements K_{ij} and $F(z_i)$ for (3), (4), and (5) are given, respectively, as

$$\left.\begin{aligned} K_{ij} &= \int_{\Delta z_j}\left(\frac{\partial^2}{\partial z^2}+k^2\right)G(z_i, z')dz' \\ F(z_i) &= -j\omega\epsilon E_z{}^i(z_i) \end{aligned}\right\} \tag{7}$$

$$\left.\begin{aligned} K_{ij} &= \int_{\Delta z_j}\left[\frac{\partial}{\partial z}G(z_i, z') + k^2\int_0^{z_i}G(\xi, z')d\xi\right]dz' \\ F(Z_i) &= -j\omega\epsilon\int_0^{z_i}E_z{}^i(\xi)d\xi + A \end{aligned}\right\} \tag{8}$$

$$\left.\begin{aligned} K_{ij} &= \int_{\Delta z_j}G(z_i, z')dz' \\ F(z_i) &= B\cos kz_i - \frac{jV}{2Z_0}\sin k\,|\,z_i\,| \end{aligned}\right\} \tag{9}$$

where Δz_j's, the subdivisions of the antenna, as shown in Fig. 1 are sufficiently small so that the current in each may be considered constant.

We notice that the integral in (7) does not converge at $i=j$. Whether the often used approximation for a thin antenna of radius a,

$$\oint_c G(z, c; z', c')dc' \approx 2\pi a\left\{\frac{e^{-jk[(z-z')^2+a^2]^{1/2}}}{4\pi[(z-z')^2+a^2]^{1/2}}\right\} \tag{10}$$

can be applied in the divergent integral of (7) is open to question [5]–[7]. An inspection of (6), (7), and (10) indicates that such approximation will not lead to the correct solution. This is so because, if approximations (7) and (10) are used in the limit of small radius a (6) approaches a diagonal matrix. That is to say, for a very thin antenna, the solution of (6) would then give $J(z)\partial E_z{}^i(z)$, which is not compatible with the well founded knowledge of antenna current distributions.

The improper integrals in (8) and (9) at $i=j$ may be integrated by using Cauchy's principal value. In these cases, we may also use the approximation (10). Actual computations based on such an approximation indeed give correct results. This possibly accounts for the fact that approximations (7) and (10) have been successfully used in variational form [8], [9], since the variational formulation introduces an additional integration, which in effect suppresses the divergent nature of the integral.

Of particular importance in the inversion of a large matrix is the problem of round-off errors accumulated through large number of arithmetic operations. In general, the round-off errors depend on the orientations of the hyperplanes represented by each row of the matrix, in the n-dimensional vector space. Qualitatively speaking, the round-off errors will be small if the hyperplanes are essentially perpendicular to one another, and the reverse is true if two or more of them are almost parallel [10]. Inspection of (8) indicates that for small radius a the coefficient K_{ij} will be small for $i<j$, and large for $i \geq j$. Hence, in the limit of a very thin antenna, the matrix elements described by (8) approach those of a triangular matrix. For the same situation, however, the matrix elements described by (9) approach those of a diagonal matrix, which is certainly superior to a triangular one in view of the previous consideration on computational errors. We shall, therefore, use integral (5) as the basis of our calculations.

A few typical results of calculation on dipole antennas are shown in Figs. 2–4. It is of interest to note that calculations based on the model of a slice generator excitation [1], and those based on the model of a magnetic loop current excitation [11] have no noticeable differences in their results.

Arbitrary Thin Wire Antennas

The extension of (3) and (4) to describe a general curved wire antenna is immediate, provided a curved cylindrical coordinate system is used. Figure 5 describes such a coordinate system, where s is the arc length measured from the feed gap, and $\hat{s}$ is the unit tangent vector at s. If the radius a of the wire is sufficiently small so that the current density may be considered to be uniform around the periphery of the wire, the corresponding integral (3) and (4) for a curved wire antenna are, respectively, [5]

$$\int_L J(s') \left[\frac{\partial^2}{\partial s \partial s'} G(s, s') - k^2 G(s, s') \hat{s} \cdot \hat{s}' \right] ds' = j\omega\epsilon E_s{}^i(s) \quad (11)$$

and

$$\int_L J(s') \left[\frac{\partial}{\partial s'} G(s, s') - k^2 \int_0^s G(\xi, s') \hat{\xi} \cdot \hat{s}' d\xi ds' \right] = j\omega\epsilon \int_0^s E_s{}^i(\xi) d\xi + A \quad (12)$$

The extension of (5) to describe a general curved wire antenna is not so apparent. The complication arises in that the kernel of the closed-cycle type is essential in the conventional way of deriving integral (5). Such a kernel has the special property

$$\frac{\partial}{\partial s} K(s, s') = - \frac{\partial}{\partial s'} K(s, s') \quad (13)$$

The structures which give rise to kernels of this type are limited to straight wires, circular arcs, and helical wires [9]. In the following we shall attempt to generalize (5) so as to include wire antennas of arbitrary geometry.

In accord with the assumptions of a thin wire antenna, the tangential component of the vector potential and scalar potential on the antenna are given, respectively, as

$$A_s(s) = \int_L J(s') G(s, s') \hat{s} \cdot \hat{s}' ds' \quad (14)$$

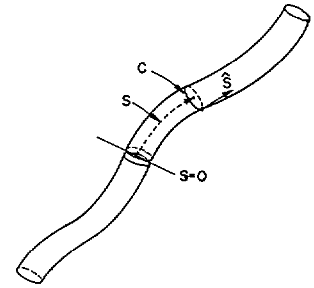

Fig. 5. A curved cylindrical coordinate system.

and

$$\phi(s) = \frac{-1}{j\omega\epsilon} \int_L \frac{dJ(s')}{ds'} G(s, s') ds' \quad (15)$$

We define a scalar function $\Phi(s)$ as

$$\Phi(s) = - j\omega\epsilon \int_0^s \phi(\xi) d\xi = \int_0^s \int_L \frac{dJ(s')}{ds'} G(\xi, s') d\xi \quad (16)$$

Integrating (16) by parts and considering $J(s)$ to vanish at both ends, we obtain

$$\Phi(s) = - \int_0^s \int_L J(s') \frac{\partial G(\xi, s')}{\partial s'} d\xi \quad (17)$$

For the s component of the electric field on the antenna to vanish, it is required that

$$E_s(s) + E_s{}^i(s) = 0 \quad (18)$$

where $E_s{}^i(s)$ is the s component of the incident electric field when the antenna is receiving, or it is the impressed field of the source if the antenna is transmitting.

From the well-known equation

$$E_s(s) = - \nabla_s \phi - j\omega\mu A_s$$

we have

$$k^2 A_s(s) - j\omega\epsilon \frac{d\phi(s)}{ds} = - j\omega\epsilon E_s{}^i(s) \quad (19)$$

or

$$\frac{d^2\Phi(s)}{ds^2} = - k^2 A_s(s) - j\omega\epsilon E_s{}^i(s) \quad (20)$$

Adding $k^2\Phi(s)$ to both sides of (20), we obtain

$$\frac{d^2\Phi(s)}{ds^2} + k^2\Phi(s) = k^2(\Phi(s) - A_s(s)) - j\omega\epsilon E_s{}^i(s) \quad (21)$$

The solution of (21) is

$$\Phi(s) = C \cos ks + D \sin k|s| + \int_0^s k(\Phi(\xi) - A_\xi(\xi)) \sin k(s - \xi) - \frac{j}{Z_0} \int_0^s E_\xi{}^i(\xi) \sin k(s - \xi) d\xi \quad (22)$$

Since $\Phi(0)=0$, we see that the constant C must vanish. Now consider the integration

$$F(s) = k\int_0^s \Phi(\xi)\sin k(s-\xi)d\xi$$
$$= -k\int_0^s \int_0^\xi \int_L J(s') \cdot \frac{\partial G(\eta, s')}{\partial s'} ds' d\eta \sin k(s-\xi)d\xi \tag{23}$$

After changing the order of integration in (23), we obtain

$$F(s) = -k\int_L \int_0^s \int_\eta^s J(s')\frac{\partial G(\eta, s')}{ds'}\sin k(s-\xi)d\xi d\eta ds'$$
$$= -\int_L\int_0^s J(s')\frac{\partial G(\eta, s')}{ds'}[1-\cos k(s-\eta)]d\eta ds'$$
$$= \Phi(s) + \int_L\int_0^s J(s')\frac{\partial G(\eta, s')}{ds'}\cos k(s-\eta)d\eta ds' \tag{24}$$

Next we consider the integration

$$H(s) = k\int_0^s A_\xi(\xi)\sin k(s-\xi)d\xi$$
$$= k\int_0^s\int_L J(s')G(\xi, s')\hat{\xi}\cdot\hat{s}'\sin k(s-\xi)d\xi ds' \tag{25}$$

Integration by parts gives

$$H(s) = \int_L J(s')G(\xi, s')\hat{\xi}\cdot\hat{s}'\cos k(s-\xi)\Big|_{\xi=0}^{\xi=s} ds'$$
$$-\int_0^s\int_L\left[\frac{\partial G(\xi, s')}{d\xi}\hat{\xi}\cdot\hat{s}' + G(\xi, s')\frac{\partial(\hat{\xi}\cdot\hat{s}')}{d\xi}\right]J(s') \cdot\cos k(s-\xi)d\xi$$
$$= \int_L J(s')G(s, s')\hat{s}\cdot\hat{s}'ds'$$
$$-\int_L J(s')G(0, s')\hat{o}\cdot\hat{s}'\cos ksds'$$
$$-\int_0^s\int_L\left[\frac{\partial G(\xi, s')}{\partial\xi}\hat{\xi}\cdot\hat{s}' + G(\xi, s')\frac{\partial(\hat{\xi}\cdot\hat{s}')}{\partial\xi}\right]J(s') \cdot\cos k(s-\xi)d\xi \tag{26}$$

Substituting (24) and (26) into (2), we obtain the integral equation for the current,

$$\int_L J(s')\pi(s, s')ds' = D\sin k|s|$$
$$+\int_L J(s')G(0, s')\hat{o}\cdot\hat{s}'\cos ksds'$$
$$-\frac{j}{Z_0}\int_0^s E_\xi^i(\xi)\sin k(s-\xi)d\xi \tag{27}$$

where

$$\pi(s, s') = G(s, s')\hat{s}\cdot\hat{s}' - \int_0^s\left[\frac{\partial G(\xi, s')}{\partial\xi}\hat{\xi}\cdot\hat{s}' + \frac{\partial G(\xi, s')}{\partial s'} + G(\xi, s')\frac{\partial(\hat{\xi}\cdot\hat{s}')}{\partial\xi}\right] \cdot\cos k(s-\xi)d\xi \tag{28}$$

The term $D\sin k|s|$ represents the effect of a slice generator which is redundant when the integral of E_ξ^i is present. Indeed, if $E_\xi^i(\xi)=V/2\delta(\xi)$, where $\delta(\xi)$ is the Dirac delta function, we have

$$\frac{-j}{Z_0}\int_0^s E_\xi^i(\xi)\sin k(s-\xi)d\xi = \frac{-jV}{2Z_0}\sin k|s| \tag{29}$$

which is consistent with (5).

To show that (27) reduces to (5) for a dipole antenna, we assume the source to be a slice generator, and notice that in this particular case

$$\frac{\partial G(\xi, s')}{\partial\xi} = -\frac{\partial G(\xi, s')}{\partial s'}$$

and

$$\hat{\xi}\cdot\hat{s} = 1$$

Hence, (27) becomes

$$\int_L J(z')G(z, z')dz'$$
$$= \int_L J(z')G(0, z')dz'\cos kz - \frac{jV}{2Z_0}\sin k|z| \tag{30}$$

Comparing (30) with (5), we have

$$B = \int_L J(z')G(0, z')dz'$$

which may be shown to be correct by considering (5) at $z=0$. Consequently, the term $\int_L J(x')G(0, z')dz'$ should be replaced by a constant, which has to be determined by the condition of the current at the ends of the antenna, otherwise the solution of the integral equation will not be unique. Therefore, the integral equation describing an arbitrary thin wire antenna is

$$\int_L J(s')\pi(s, s')ds'$$
$$= C'\cos ks - \frac{j}{Z_0}\int_0^s E_\xi^i(\xi)\sin k(s-\xi)d\xi \tag{31}$$

The specialization of (31) to a circular loop antenna also agrees with that derived by Adachi [12].

A further check of the integral equation may be effected as following. We differentiate (31) twice with respect to s, and make use of the differential relation,

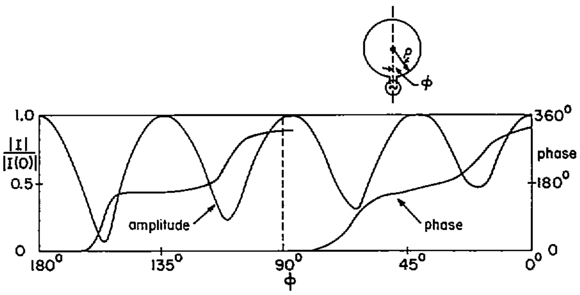

Fig. 6. Amplitude and phase of the current on a circular loop antenna of the parameter $\Omega=2 \log 8/a\rho=15$ (ρ=radius of the loop, a=radius of wire) $k\rho=4.0$.

$$\frac{d}{dx}\int_0^{f(x)} g(x, x')dx' = \int_0^{f(x)} \frac{\partial}{\partial x} g(x, x')dx' + g(x, x')\big|_{x'=x}f'(x) \quad (32)$$

and obtain

$$k^2 \int_L \int_0^s J(s')\left[\frac{\partial G(\xi, s')}{\partial \xi}\,\hat{\xi}\cdot\hat{s}' + \frac{\partial G(\xi, s')}{\partial s'} + G(\xi, s')\,\frac{\partial(\hat{\xi}\cdot\hat{s}')}{\partial \xi}\right] \cdot \cos k(s-\xi)d\xi ds' - \frac{\partial}{\partial s}\int_L J(s')\,\frac{\partial G(s, s')}{\partial s'}\,ds' = -k^2C' \cos ks + j\,\frac{k^2}{Z_0}\int_0^s E_\xi{}^i(\xi) \cdot \sin k(s-\xi)d\xi - j\omega\epsilon E_\xi{}^i(s) \quad (33)$$

Multiplying (31) by k^2 and adding the result to (33), results in

$$k^2 \int_L J(s')G(s, s')\hat{s}\cdot\hat{s}'ds' - \frac{\partial}{\partial s}\int_L J(s')\,\frac{\partial G(s, s')}{\partial s'}\,ds = -j\omega\epsilon E_s{}^i(s) \quad (34)$$

which is essentially (19). Therefore, the integral (31) is shown to be the correct one.

Applications

Equation (31) has been applied to circular loop antennas [13] and equiangular spiral antennas. The representative results are shown in Figs. 6 and 7.

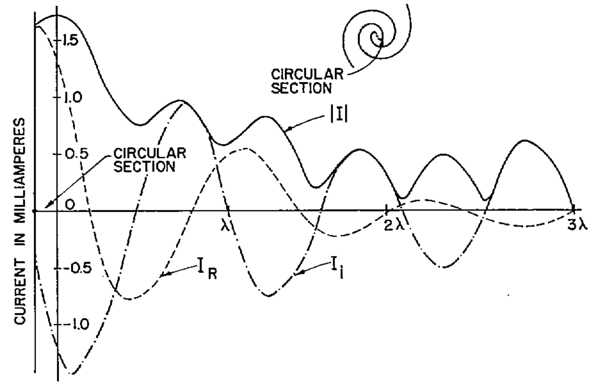

Fig. 7. Current distribution of a 3λ-arm equiangular spiral antenna or $r=ce^{a\phi}$, with $a=0.2$, $c=0.05\lambda$, radius of wire 0.025λ.

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