without degrading the pattern near θ_M. An equivalent procedure would be to relocate the fixed zeros.

C. V. BAKER[1]
Goodyear Aerospace Corp.
Litchfield Park, Ariz.

REFERENCES

[1] R. F. Hyneman and R. M. Johnson, "A technique for the synthesis of shaped-beam radiation patterns with approximately equal-percentage ripple," *IEEE Trans. Antennas and Propagation*, vol. AP-15, pp. 736–743, November 1967.
[2] S. A. Schelkunoff, "A mathematical theory of linear arrays," *Bell Sys. Tech. J.*, vol. 22, pp. 80–107, January 1943.

[1] Now with the General Research Corporation, Santa Barbara, Calif.

A Trigonometric Approximation to the Current in the Solution of Hallen's Equation

Abstract—**A simple trigonometric expansion for the current in the solution of Hallen's equation gives a good approximation to the impedance of isolated thin cylindrical dipole antennas. For the half-wavelength dipole a two-term representation for the current is found to give an impedance which compares favorably with experimental measurements.**

If a dipole antenna is viewed as a thin perfectly conducting cylinder, the first-order integral equation which describes the axial-current distribution may be expressed in the form [1]

$$\eta_0 \int_{-h}^{+h} I_z(z')K(z,z')\,dz' = -j\,4\pi(C_1 \cos kz + (V_0/2)\sin k|z|) \quad (1)$$

where

η_0 = impedance of medium
V_0 = applied base-gap voltage
$I_z(z')$ = axial current
C_1 = constant determined by electric field off end of antenna

and

$$K(z,z') = \frac{\exp\left[-jk\sqrt{(z-z')^2+a^2}\right]}{\sqrt{(z-z')^2+a^2}}. \quad (2)$$

In the expression for the kernel K, a is the radius of the dipole and k is the wavenumber. In free space the intrinsic impedance may be approximated by 120π ohms. Using this approximation, the integral equation may be rewritten as

$$\int_{-h}^{+h} I_z(z')K(z,z')\,dz' = -j\,C_2 \cos kz - j\frac{V_0}{60}\sin k|z|. \quad (3)$$

To obtain an approximate expression for the current, assume that it may be expanded in the following form:

$$I_z(z') = \sum_{n=1;\,n\text{ odd}}^{m} a_n \cos(n\pi z'/2h) + \sum_{n=2;\,n\text{ even}}^{m} b_n \sin(n\pi|z'|/2h). \quad (4)$$

Note that this expansion is an even function of z'; therefore, this form of expansion is limited to axial currents which are evenly distributed along the length of the antenna. The form of the current shown in (4) differs from those used by Duncan and Hinchey [2] and Storm [3] because of the addition of the sine terms. The addition of the sine terms is necessary, as will be shown in the subsequent treatment of the half-wavelength antenna. The expression for the current (4) may be written equivalently as

$$I_z(z') = \sum_{n=1}^{m} B_n \sin(n\pi/2h)(h-|z'|). \quad (5)$$

If (4) is substituted into the left-hand side of (3) it takes the form

$$\sum_{n=1;\,n\text{ odd}}^{m} a_n \int_{-h}^{+h} \cos\frac{n\pi z'}{2h} K(z,z')\,dz' + \sum_{n=2;\,n\text{ even}}^{m} b_n \int_{-h}^{+h} \sin\frac{n\pi|z'|}{2h} K(z,z')\,dz' = -j\,C_2\cos kz - j\,(V_0/60)\sin k|z| \quad (6)$$

where the order of summation and integration has been interchanged. Equation (6) has $m+1$ unknowns; letting z take on $m+1$ values (6) becomes a set of $m+1$ linear equations.

It is useful to make the following definitions:

$$c_n(z,h) = \int_{-h}^{+h} \cos\frac{n\pi z'}{2h} K(z,z')\,dz', \quad n \text{ odd}$$

$$s_n(z,h) = \int_{-h}^{+h} \sin\frac{n\pi|z'|}{2h} K(z,z')\,dz', \quad n \text{ even}. \quad (7)$$

Substituting (7) into (6) gives the following matrix equation:

$$[A] = [F]\cdot[X] \quad (8)$$

where

$$[A] = -j\frac{V_0}{60}\begin{bmatrix} \sin k|z_1| \\ \sin k|z_2| \\ \vdots \\ \sin k|z_{m+1}| \end{bmatrix} \quad (9)$$

$$[F] = \begin{bmatrix} c_1(z_1,h) & s_2(z_1,h) & c_3(z_1,h)\cdots w_m & j\cos kz_1 \\ c_1(z_2,h) & & & \\ \vdots & & & \\ c_1(z_{m+1},h) & & \cdots & j\cos kz_{m+1} \end{bmatrix} \quad (10)$$

$$[X] = \begin{bmatrix} a_1 \\ b_2 \\ a_3 \\ \vdots \\ C_2 \end{bmatrix}. \quad (11)$$

If m is even, $w_m = s_m(z_1,h)$; if m is odd, $w_m = c_m(z_1,h)$. The elements of the $[F]$ and $[A]$ matrices may easily be computed on a high-speed digital computer. Therefore, the coefficients in the expansion for the current are determined by the equation

$$[X] = [F]^{-1}\cdot[A]. \quad (12)$$

The half-wavelength dipole has been analyzed using this technique. The $m+1$ values of z needed to write the system of linear equations are selected by choosing uniformly spaced points along the half-length of the antenna h. The impedance of the antenna is determined by dividing the driving voltage by the current at the center ($z = 0$). Using (4) the impedance is

$$Z = V_0 / \sum_{n=1;\,n\text{ odd}}^{m} a_n. \quad (13)$$

Fig. 1 shows the variation of the impedance of a half-wavelength dipole of radius $h/a = 74.207$. For the lowest approximation $m = 1$, the classical impedance $73 + j41$ ohms is obtained. Note that in (4), if $m = 1$, the current is approximated by a single cosine,

Manuscript received February 20, 1969.

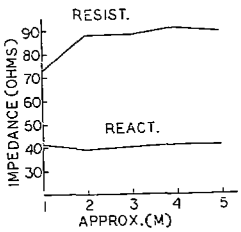

Fig. 1. Variation of dipole impedance with degree of approximation, $h = \lambda/4$, $h/a = 74.207$.

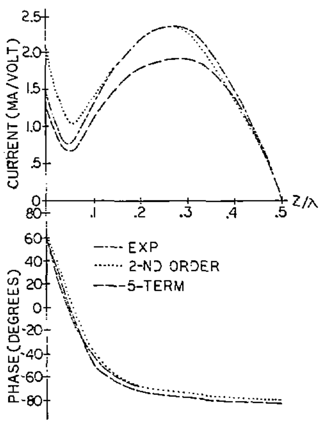

Fig. 2. Current distribution of full-wavelength antenna, $h/a = 74.207$.

which is just the zero-order current obtained from

$$I_0(z) = I_m \sin k(h - |z|) \tag{14}$$

for a half-wavelength antenna. Increasing the approximation to $m = 2$ is equivalent to adding a sine term to the expansion. This causes a significant improvement in the impedance, as would be expected. Experimental measurements indicate that for $z > 0$ the slope of the current is positive at the center of the antenna and the addition of the sine term provides the necessary slope. As Fig. 1 indicates, increasing the expansion to $m = 5$ results in the impedance converging to $89.6 + j41.1$ ohms. For the same antenna King [4] gives a third-order resistance of 89.9 and a second-order reactance of 41.7 from the King–Middleton theory. The impedance obtained using two terms is $88.0 + j39.2$, which in itself demonstrates remarkable agreement with more accurate values.

The method is presently being applied to antennas of arbitrary length. Fig. 2 is a graph of the current distribution using a five-term approximation for the current. A curve showing the experimentally measured current is included as a reference, as is the second-order distribution obtained by King [4]. The trigonometric expansion appears to give a reasonable approximation to the current distribution.

N. P. Neff
C. A. Siller, Jr.
J. D. Tillman, Jr.
Dept. of Elec. Engrg.
University of Tenn.
Knoxville, Tenn.

References

[1] R. W. P. King, "Dipoles in dissipative media," in *Electromagnetic Waves*, R. E. Langer, Ed. Madison, Wis.: University of Wisconsin Press, 1962, pp. 199–241.
[2] R. H. Duncan and F. A. Hinchey, "Cylindrical antenna theory," *J. Res. NBS*, sec. D (Radio Propagation), vol. 64D, pp. 569–584, September 1960.
[3] B. Storm, "Cylindrical aerials," *Wireless Engineer*, vol. 29, pp. 174–176, July 1952.
[4] R. W. P. King, *The Theory of Linear Antennas*. Cambridge, Mass.: Harvard University Press, 1956.

Transient Response of a Dipole over a Circular Ground Screen

Abstract—The time-domain response of a vertical electric dipole located over a circular ground screen is considered. The relevant time-harmonic solution is used as a starting point. By making a number of approximations a very simple formula is obtained for the transient response of the far field when the dipole current moment is a ramp function of time. It is shown that for early times, the pattern response appears as if the ground screen were of infinite extent, while at very long times, the response approaches that expected for a homogeneous flat ground.

Introduction

Low-angle radiation from a dipole antenna over a finitely conducting ground may be improved by the use of a metallic ground screen. The limited size of any practical system does not permit one to realize fully the ideal pattern for a perfectly conducting ground plane. Quantitative studies of this important effect have been the subject of various theoretical and experimental investigations [1]–[3]. Without exception the analyses have been carried out for time-harmonic fields. In this communication we wish to develop a transient solution. To simplify the discussion, a relatively simple model is adopted, and the source dipole moment is assumed to be a step function of time.

Formulation of the Time-Harmonic Problem

With respect to a cylindrical coordinate system (ρ,ϕ,z) the earth's surface is at $z = 0$, and the source electric dipole is located at $z = h$ on the axis. As indicated in Fig. 1, the circular ground screen is located on the surface $z = 0$ and is bounded by $\rho = a$. In the absence of the ground system the surface impedance is Z, which does not vary with ρ. On the other hand, the surface impedance of the ground screen, in parallel with the homogeneous earth, is Z_a'. This, in effect, is saying that the effective surface impedance for the entire ground plane is Z_a' for $0 < \rho < a$, and Z for $\rho > a$.

For convenience, we will discuss the problem initially for a time factor $\exp(i\omega t)$. The current moment of the dipole is designated $P_0(\omega)$, and the magnetic field in the far zone has the form

$$H_\phi(\omega) = \frac{i\omega P_0(\omega)}{2\pi c R_0} \exp(-i\omega R_0/c) \cos\psi_0 W'(\omega) \tag{1}$$

where R_0 is the distance to the observer, ψ_0 the angle subtended by R_0 and the earth's surface, and c the velocity of light in a vacuum. W' is a pattern function which is normalized to be unity if the dipole height h were zero, and if the entire ground plane were perfectly conducting.

An approximate formula for W' is given as follows [4]:

$$W' \simeq (1 + R_v)[G_0(h) + \Omega]/2 \tag{2}$$

where

$$R_v = \frac{\sin\psi_0 - (Z/\eta_0)}{\sin\psi_0 + (Z/\eta_0)} \tag{3}$$

$$G_0(h) = \frac{\exp[i\omega(h/c)\sin\psi_0] + R_v \exp[-i\omega(h/c)\sin\psi_0]}{1 + R_v} \tag{4}$$

and

$$\Omega = \frac{\omega}{c\cos\psi_0} \int_{\rho=0}^{a} \exp(-i\omega r_q/c) \frac{\rho^2}{r_q^2}\left(1 + \frac{c}{i\omega r_q}\right) \cdot J_1\left(\frac{\omega\rho}{c}\cos\psi_0\right)\left(\frac{Z_a' - Z}{\eta_0}\right) d\rho \tag{5}$$

Manuscript received February 27, 1969; revised June 9, 1969. This work was supported by Advanced Research Projects Agency under ARPA Order 932.