We can also predict rotations of polarization upon incidence at, and transmission through, the second boundary $A'C'$ (Fig. 2). In like fashion, these "exit Kerr angles," δ_3 and δ_4, can be shown to be the negative of the corresponding "entrance Kerr angles" δ_1 and δ_2. In short,

$$\delta_1 = -\delta_3 = 1/2(\phi_L' - \phi_R') \tag{22}$$

$$\delta_2 = -\delta_4 = 1/2(\phi_L'' - \phi_R'') \tag{23}$$

$$\delta_5 = 1/2(\beta_R - \beta_L)\cdot z. \tag{24}$$

IV. Conclusion

Summarizing, this treatment establishes

1. A quantitative description of the Kerr magneto-optical effect.

2. The existence of five angles of rotation of polarization in the complete passage of electromagnetic radiation through a magneto optically active material:

δ_1, upon reflection from the first interface
δ_2, upon transmission across the first interface
δ_3, upon reflection from the second interface
δ_4, upon transmission across the second interface
δ_5, during propagation in the active medium
δ_5 and δ_1 being, respectively, the angles of Faraday and Kerr.

3. The dependence of δ_1, δ_2, δ_3, and δ_4 upon the material constants of the media immediately adjacent to the entering and exit surfaces of the active material.

A Method for Calculating the Current Distribution of Tschebyscheff Arrays*

DOMENICK BARBIERE†

Summary—**Dolph has derived an optimum current distribution for equispaced broadside arrays based upon the properties of the Tschebyscheff polynomials.[1] Design curves are given for arrays of 8, 12, 16, 20, and 24 elements. The equations to be computed are bulky, however, so that the numerical calculations become cumbersome for arrays of more than 24 elements. In this paper, the equations of Dolph's method are considerably simplified by algebraic means with no loss in exactness. The final current expressions are given in a closed, exact form. It is also shown that the expressions for the current elements may be easily tabulated. A table for a 24-element array is constructed as an example which may readily be extended to arrays of any number of elements.**

Discussion

A BRIEF REVIEW of Dolph's derivation will first be given. It is shown that the "Tschebyscheff current distribution" may be calculated after either the side-lobe level or the position of the first null is specified. The "Tschebyscheff pattern" resulting from this current distribution is optimum in the sense that (a) if the side-lobe level is specified, the beamwidth of the resultant pattern can be proved a minimum, or (b) if the beamwidth is specified, the side-lobe level will be a minimum. A detailed calculation of the pattern is unnecessary since the character of the pattern, in particular the side-lobe and null positions, is completely specified from the well-known properties of the Tschebyscheff polynomials. In a later paper, Riblet[2] extended Dolph's method to remove some of its limitations.

It is well known that the radiation pattern of a linear equispaced broadside symmetric array of point sources is proportional to

$$\left| E_{2N-1}(\theta) \right| = \left| \sum_{k=1}^{N} I_k \cos\left[\frac{2k-1}{2}\left(\frac{2\pi d}{\lambda}\right)\sin\theta\right]\right| \tag{1}$$

$$\left| E_{2N}(\theta) \right| = \left| \sum_{k=0}^{N} I_k \cos\left[k\left(\frac{2\pi d}{\lambda}\right)\sin\theta\right]\right|, \tag{1'}$$

where (1) and (1′) apply to an even number ($2N$) and an odd number ($2N+1$) of elements, respectively. The variable θ denotes the angle between the direction of the field to the distant point P and the normal to the array, d is the element spacing, and I_k represents the current in the kth element from the center of the array. The above equations are valid only if all the currents are in phase along the array. An extension of the method for out of phase currents is given by Riblet.[2]

The odd and even cases were developed simultaneously by Dolph. However, the equations for both cases are fundamentally similar, differing essentially in the matter of superscripts and subscripts. In the section of this paper reviewing Dolph's material, therefore, the even case only will be discussed.

Introduction of the new variable

$$u = \frac{\pi d \sin\theta}{\lambda}$$

simplifies (1) to

$$F_{2N-1}(u) = \sum_{k=1}^{N} I_k \cos(2k-1)u, \tag{2}$$

where, henceforth, only the absolute values of all pattern expressions will be considered so that the absolute value signs may be omitted.

A term of the form $\cos nu$ may be expanded into a polynomial in powers of $\cos u$ whenever n is an integer. More exactly, it can be verified that

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[1] C. L. Dolph, "A current distribution which optimizes the relationship between beam-width and side-lobe level," Proc. I.R.E., vol. 34, pp. 335–348; June, 1946.

[2] H. J. Riblet and C. L. Dolph, Discussion on "A current distribution for broadside arrays which optimizes the relationship between beam-width and side-lobe level," Proc. I.R.E., vol. 35, pp. 489–492; May, 1947.

$$\cos(2k-1)u = \sum_{q=1}^{k} A_{2q-1}^{2k-1} x^{2q-1}, \tag{3}$$

where

$$A_{2q-1}^{2k-1} = (-1)^{k-q} \sum_{p=k-q}^{k} \binom{p}{p-k+q} \binom{2k-1}{2p}$$

$$x = \cos u \text{ and } \binom{n}{m} = \frac{n!}{m!(n-m)!}.$$

When (3) is substituted into (2) and the summation signs rearranged, the pattern equation, $F_{2N-1}(u)$, takes the polynomial form

$$G_{2N-1}(x) = \sum_{q=1}^{N} \sum_{k=q}^{N} I_k A_{2q-1}^{2k-1} x^{2q-1}, \tag{4}$$

where x is restricted to $|x| = |\cos u| \leqq 1$.

It will now be shown that with suitable values of the currents (I_k) the antenna pattern described by the polynomial (4) may be made to coincide with the pattern of an appropriate Tschebyscheff polynomial, which in turn possesses all of the previously mentioned optimum properties. The nonnormalized Tschebyscheff polynomials are defined by

$$T_n(z) = \cos(n \arccos z); \quad |z| \leqq 1, \tag{5}$$

where n is an integer. Clearly, the maxima and nulls of (5) are given by

$$|T_n(z)| = 1 \quad \text{for} \quad z = \cos\frac{k\pi}{n}; \; k = 0, 1, 2, \cdots, n \tag{6}$$

$$T_n(z) = 0 \quad \text{for} \quad z = \cos(2k-1)\frac{\pi}{2n};$$

$$k = 1, 2, \cdots, n.$$

$T_n(z)$ is also of the form $\cos n\phi$, where $\phi = \arccos z$ and n is an integer. Therefore, it may be converted into a polynomial in powers of $\cos\phi = \cos(\arccos z) = z$. Expansion of $T_{2N-1}(z)$ using (3) yields

$$T_{2N-1}(z) = \cos[(2N-1)\arccos z]$$

$$= \sum_{q=1}^{N} A_{2q-1}^{2N-1} z^{2q-1}. \tag{7}$$

Forms (5) and (7) of the Tschebyscheff polynomial are equivalent. Whenever $T_n(z)$ is expressed in the finite polynomial form (7), the limits of z may be extended to $\pm\infty$. In the region $|z| \leqq 1$ forms (5) and (7) yield the same results, with (5) simpler for computational purposes. However, for $|z| \geqq 1$, the polynomial form only is valid. Equations (7) and (4) are similar in form, except that while $|x| = |\cos u| \leqq 1$ in (4), the limits of z above are $\pm\infty$. Nevertheless, the two polynomials may be made to correspond exactly by restricting the variable in (7) to $z \leqq z_0$, where z_0 is an arbitrary parameter, and setting $x = \cos u = z/z_0$.

Equation (7) may now be written

$$T_{2N-1}(z_0 x) = \sum_{q=1}^{N} A_{2q-1}^{2N-1} z_0^{2q-1} x^{2q-1}, \tag{8}$$

where $|x| \leqq 1$. Equations (8), representing the Tschebyscheff polynomial limited to the region within $\pm z_0$, and (4), representing the antenna pattern, are now in the same form. Corresponding coefficients may be equated and solved for the currents. Thus

$$\sum_{k=q}^{N} I_k A_{2q-1}^{2k-1} = A_{2q-1}^{2N-1} z_0^{2q-1}; \quad q = 1, 2, \cdots, N, \tag{9}$$

whence

$$I_q = \frac{1}{A_{2q-1}^{2q-1}} \left\{ A_{2q-1}^{2N-1} z_0^{2q-1} - \sum_{k=q+1}^{N} I_k A_{2q-1}^{2k-1} \right\}. \tag{10}$$

If the I's are computed from (10), the resultant field pattern given by (4) will agree with the Tschebyscheff pattern shown in (8). The side-lobes and nulls of the antenna pattern will coincide with the maxima and minima of the Tschebyscheff pattern given by (6) and will occur in the region $|z_0 x| \leqq 1$. In the region $1 \leqq |z_0 x| \leqq z_0$, the Tschebyscheff polynomial rises very steeply. This portion will represent the main lobes whose shape may be deduced from the polynomial form of $T_n(z_0 x)$. It was proven rigorously by Dolph that the Tschebyscheff pattern yields a minimum beamwidth when the side-lobe levels are known and a minimum side-lobe level when the beamwidth is specified.

The adjustable parameter z_0 may be calculated when either the side-lobe level or the beamwidth (position of the first null) is given. In the first case, z_0 must satisfy the equation $T_{2N-1}(z_0) = r$, where $r/1$ is the specified main-beam to side-lobe ratio. Since $r > 1$, z_0 must be evaluated from the polynomial form of $T_{2N-1}(z_0)$. However, Dolph[2] derived a simpler formula for computing z_0 from still another form of the Tschebyscheff polynomial. The final result is

$$z_0 = \tfrac{1}{2}\left\{(r + \sqrt{r^2-1})^{1/(2N-1)} + (r - \sqrt{r^2-1})^{1/(2N-1)}\right\}. \tag{11}$$

From (6), the nulls of $T_{2N-1}(z_0 x)$ are at

$$z_0 x = \cos\left(\frac{2k-1}{2}\,\frac{\pi}{2N-1}\right),$$

whence

$$z_0 x_1^0 = \cos\frac{\pi}{2(2N-1)}$$

defines the position of the first null. When θ_0 is specified as the angular position of the first null, z_0 may be deduced from the relations

$$z_0 = \frac{1}{x_1^0}\cos\frac{\pi}{2(2N-1)};$$

$$x_1^0 = \cos u_1^0 = \cos\left(\frac{\pi d}{\lambda}\sin\theta_0\right).$$

It is hoped the short synopsis of Dolph's material presented above is sufficient so the remainder of this paper may be understood. For a more detailed account of the method, the reader is referred to the original sources.

It is evident that the numerical work involved in calculating the current distribution from (10) and z_0 from (11) can become extremely tedious as the number of elements increases. In the discussion that follows, a simple method of computing the I's is derived whereby the currents will be given immediately in simple terms of z_0. Also, a very rapid method for computing z_0 will be given.

We may equate (2) to (8) so that

$$\sum_{k=1}^{N} I_k \cos (2k-1)u = \sum_{q=1}^{N} A_{2q-1}^{2N-1} z_0^{2q-1} (\cos u)^{2q-1} \tag{12}$$

$$\sum_{k=0}^{N} I_k \cos (2ku) = \sum_{q=0}^{N} A_{2q}^{2N} z_0^{2q} (\cos u)^{2q}, \tag{12'}$$

where the odd case has been reintroduced. The left sides of the above equations may be treated as parts of a Fourier series. Thus, if both sides of (12) and (12′) are multiplied by $\cos (2k-1)u$ and $\cos (2k)u$, respectively, and integrated from 0 to 2π, there remains only

$$\pi I_k = \sum_{q=1}^{N} A_{2q-1}^{2N-1} z_0^{2q-1} \int_0^{2\pi} \cos^{2q-1} u \cos (2k-1) u du \tag{13}$$

$$\pi I_k = \sum_{q=0}^{N} A_{2q}^{2N} Z_0^{2q} \int_0^{2\pi} \cos^{2q} u \cos (2ku) du. \tag{13'}$$

The integrals above are of the form

$$\int_0^{2\pi} \cos^m x \cos nx dx,$$

where m and n are arbitrary integers. Using formulas 360 and 267 of Pierce's tables, it can be seen that

$$\int_0^{2\pi} \cos^m x \cos nx dx$$
$$= 2\pi \left\{ \left(\frac{m}{m+n}\right)\left(\frac{m-1}{m+n-2}\right) \cdots \left(\frac{m-n+1}{m-n+2}\right)\left(\frac{m-n-1}{m-n}\right) \cdots (\tfrac{1}{2}) \right\}, \tag{14}$$

with the restriction that $m \geqq n$ and $m-n$ be even. When $n > m$ or $m-n$ is odd, the integral is equal to zero. Setting $m = 2q-1$, $n = 2k-1$ for (12) and $m = 2q$, $n = 2k$ for (12′) yields for the currents

$$I_k = \sum_{q=k}^{N} A_{2q-1}^{2N-1} z_0^{2q-1} \left\{ 2\left(\frac{2q-1}{2q+2k-2}\right)\left(\frac{2q-2}{2q+2k-4}\right) \cdots \left(\frac{2q-2k+1}{2q-2k+2}\right)\left(\frac{2q-2k-1}{2q-2k}\right) \cdots (\tfrac{1}{2}) \right\} \tag{15}$$

$$I_k = \sum_{q=k}^{N} A_{2q}^{2N} z_0^{2q} \left\{ 2\left(\frac{2q}{2q+2k}\right)\left(\frac{2q-1}{2q+2k-2}\right) \cdots \left(\frac{2q-2k+1}{2q-2k+2}\right)\left(\frac{2q-2k-1}{2q-2k}\right) \cdots (\tfrac{1}{2}) \right\}, \tag{15'}$$

where $m \geqq n (q \geqq k)$.

If the numerator of the bracketed expression in (15) is multiplied by $(2q-2k)(2q-2k-2) \cdots 2 = (q-k)!2^{q-k}$, it then becomes equal to $2(2q-1)!$ Thus, the numerator in question may be written as $(2q-1)!/(q-k)!2^{q-k-1}$. The denominator of this expression may be written as $1/(q+k-1)!2^{q+k-1}$. The substitution of the above into (15) and the application of a similar manipulation to the bracketed expression of (15′) yields

$$I_k = \sum_{q=k}^{N} A_{2q-1}^{2N-1} z_0^{2q-1} \frac{(2q-1)!}{(q-k)!(q+k-1)!2^{2q-2}} \tag{16}$$

$$I_k = \sum_{q=k}^{N} A_{2q}^{2N} z_0^{2q} \frac{(2q)!}{(q-k)!(q+k)!2^{2q-1}}, \tag{16'}$$

where (16) is for an even number ($2N$) elements and (16′) for an odd number ($2N+1$) elements.

Further simplification is still possible by incorporating the A's, or Tschebyscheff coefficients (see (7)), into the factorial expression. The Tschebyscheff polynomial is actually a particular form of Gauss's hypergeometric series and may be written[3]

$$T_n(x) = 2^{2n-1}\left\{ x^n - \frac{n}{1!2^2} x^{n-2} + \frac{(n)(n-3)}{2!2^4} x^{n-4} - \frac{(n)(n-4)(n-5)}{3!2^6} x^{n-6} + \cdots \right\}. \tag{17}$$

With some manipulation, it can be seen that $T_n(x)$ may also be written in the alternate form

$$T_n(x) = \sum_{m=0,1}^{n} (-1)^{(n-m)/2} 2^{2n-1} \left\{ \frac{n\left(m + \frac{n-m}{2} - 1\right)!}{\left(\frac{n-m}{2}\right)! m! 2^{n-m}} x^m \right\}, \tag{18}$$

where $m = n, n-2, n-4 \cdots 0$, or 1. By comparing (18) with (7), the Tschebyscheff coefficients become

$$A_{2q-1}^{2N-1} = (-1)^{N-q} 2^{2N-2} \frac{(2N-1)(q+N-2)!}{(N-q)!(2q-1)!2^{2N-2q}} \tag{19}$$

$$A_{2q}^{2N} = (-1)^{N-q} 2^{2N-1} \frac{(2N)(q+N-1)!}{(N-q)!(2q)!\,2^{2N-2q}}. \tag{19'}$$

Introducing the results (19) and (19′) into (16) and (16′) yields finally

$$I_k = \sum_{q=k}^{N} (-1)^{N-q} z_0^{2q-1} \frac{(2N-1)(q+N-2)!}{(q-k)!(q+k-1)!(N-q)!} \tag{20}$$

$$I_k = \sum_{q=k}^{N} (-1)^{N-q} z_0^{2q} \frac{(2N)(q+N-1)!}{(q-k)!(q+k)!(N-q)!}, \tag{20'}$$

where (20) and (20′) apply to $2N$ and $2N+1$ element arrays, respectively.

Equations (20) and (20′) are most readily solved by

[3] H. Margenau and G. Murphy, "Mathematics of Physics and Chemistry," Van Nostrand and Co., Inc., New York, N. Y., p. 74; 1943. This equation is given incorrectly in the earlier editions of the book.

constructing a table of coefficients wherein the I_k's form the rows and the z_0's the columns. See Table I for 24 elements. The 24-element case was solved by Dolph, and is introduced here as a check on the new method of attack. The results agree perfectly with those of Dolph, and yet were obtained with only a few hours of computation. The table shows that once the upper left-hand corner is constructed by substituting for q, N, and k the remainder of the table may be readily extended from sequence considerations.

The tables also lend themselves to quick checks for errors. The sums across the rows, which are actually the current values for $z_0=1$, should equal zero for all the currents except the final one, I_N. Thus, for the even case,

$$\sum_{k=1}^{N} I_k(z_0)\cos(2k-1)u = T_{2N-1}(z_0x).$$

But, when $z_0=1$, $T_{2N-1}(z_0x)=\cos\left[(2N-1)\arccos x\right]=\cos(2N-1)u$. Thus, $I_k=0$, except for $k=N$ where $I_N=1$. Also, the sum down each column should equal the coefficient of the corresponding term of the Tschebyscheff polynomial. When $x=\cos u=1$, $\cos(2k-1)u=1$ for all k so that $\sum I_k(z_0)=T_{2N-1}(z_0)$ from the above.

TABLE I
CURRENT DISTRIBUTION FOR A 24-ELEMENT ARRAY YIELDING A TSCHEBYSHEFF PATTERN

	$q=12(+z_0^{23})$	$q=11(-z_0^{21})$	$q=10(+z_0^{19})$	$q=9(-z_0^{17})$	$q=8(+z_0^{15})$	$q=7(-z_0^{13})$	$q=6(+z_0^{11})$	$q=5(-z_0^{9})$	$q=4(+z_0^{7})$	$q=3(-z_0^{5})$	$q=2(+z_0^{3})$	$q=1(-z_0)$
I_1	$\frac{(23)22!}{12!11!0!}$ 1352078	$\frac{(23)21!}{11!10!1!}$ 8112468	$\frac{(23)20!}{10!9!2!}$ 21246940	$\frac{(23)19!}{9!8!3!}$ 31870410	$\frac{(23)18!}{8!7!4!}$ 30193020	$\frac{(23)17!}{7!6!5!}$ 18786768	$\frac{(23)16!}{6!5!6!}$ 7735728	$\frac{(23)15!}{5!4!7!}$ 2072070	$\frac{(23)14!}{4!3!8!}$ 345345	$\frac{(23)13!}{3!2!9!}$ 32890	$\frac{(23)12!}{2!1!10!}$ 1518	$\frac{(23)11!}{1!0!11!}$ 23
I_2	$\frac{(23)22!}{13!10!0!}$ 1144066	$\frac{(23)21!}{12!9!1!}$ 6760390	$\frac{(23)20!}{11!8!2!}$ 17383860	$\frac{(23)19!}{10!7!3!}$ 25496328	$\frac{(23)18!}{9!6!4!}$ 23483460	$\frac{(23)17!}{8!5!5!}$ 14090076	$\frac{(23)16!}{7!4!6!}$ 5525520	$\frac{(23)15!}{6!3!7!}$ 1381380	$\frac{(23)14!}{5!2!8!}$ 207207	$\frac{(23)13!}{4!1!9!}$ 16445	$\frac{(23)12!}{3!0!10!}$ 506	
	$\frac{(23)22!}{14!9!0!}$ 817190	$\frac{(23)21!}{13!8!1!}$ 4680270	$\frac{(23)20!}{12!7!2!}$ 11589240	$\frac{(23)19!}{11!6!3!}$ 16224936	$\frac{(23)18!}{10!5!4!}$ 14090076	$\frac{(23)17!}{9!4!5!}$ 7827820	$\frac{(23)16!}{8!3!6!}$ 2762760	$\frac{(23)15!}{7!2!7!}$ 592020	$\frac{(23)14!}{6!1!8!}$ 69069	$\frac{(23)13!}{5!0!9!}$ 3289		
I_4	$\frac{(23)22!}{15!8!0!}$ 490314	$\frac{(23)21!}{14!7!1!}$ 2674440	$\frac{(23)20!}{13!6!2!}$ 6240360	$\frac{(23)19!}{12!5!3!}$ 8112468	$\frac{(23)18!}{11!4!4!}$ 6404580	$\frac{(23)17!}{10!3!5!}$ 3131128	$\frac{(23)16!}{9!2!6!}$ 920920	$\frac{(23)15!}{8!1!7!}$ 148005	$\frac{(23)14!}{7!0!8!}$ 9867			
I_5	$\frac{(23)22!}{16!7!0!}$ 245157	$\frac{(23)21!}{15!6!1!}$ 1248072	$\frac{(23)20!}{14!5!2!}$ 2674440	$\frac{(23)19!}{13!4!3!}$ 3120180	$\frac{(23)18!}{12!3!4!}$ 2134860	$\frac{(23)17!}{11!2!5!}$ 853944	$\frac{(23)16!}{10!1!6!}$ 184184	$\frac{(23)15!}{9!0!7!}$ 16445				
I_6	$\frac{(23)22!}{17!6!0!}$ 100947	$\frac{(23)21!}{16!5!1!}$ 468027	$\frac{(23)20!}{15!4!2!}$ 891480	$\frac{(23)19!}{14!3!3!}$ 891480	$\frac{(23)18!}{13!2!4!}$ 492660	$\frac{(23)17!}{12!1!5!}$ 142324	$\frac{(23)16!}{11!0!6!}$ 16744					
I_7	$\frac{(23)22!}{18!5!0!}$ 33649	$\frac{(23)21!}{17!4!1!}$ 137655	$\frac{(23)20!}{16!3!2!}$ 222870	$\frac{(23)19!}{15!2!3!}$ 178296	$\frac{(23)18!}{14!1!4!}$ 70380	$\frac{(23)17!}{13!0!5!}$ 10948						
I_8	$\frac{(23)22!}{19!4!0!}$ 8855	$\frac{(23)21!}{18!3!1!}$ 30590	$\frac{(23)20!}{17!2!2!}$ 39330	$\frac{(23)19!}{16!1!3!}$ 22287	$\frac{(23)18!}{15!0!4!}$ 4692							
I_9	$\frac{(23)22!}{20!3!0!}$ 1771	$\frac{(23)21!}{19!2!1!}$ 4830	$\frac{(23)20!}{18!1!2!}$ 4370	$\frac{(23)19!}{17!0!3!}$ 1311								
I_{10}	$\frac{(23)22!}{21!2!0!}$ 253	$\frac{(23)21!}{20!1!1!}$ 483	$\frac{(23)20!}{19!0!2!}$ 230									
I_{11}	$\frac{(23)22!}{22!1!0!}$ 23	$\frac{(23)21!}{21!0!1!}$ 23										
I_{12}	$\frac{(23)22!}{23!0!0!}$ 1											
$T_{23}(z_0)$	4194304	24117248	60293120	85917696	76873728	44843008	17145856	4209920	631488	52624	2024	23

$$I_k=\sum_{q=k}^{N}(-1)^{N-q}z_0^{2q-1}\frac{(2N-1)(q+N-2)!}{(q+k-1)!(q-k)!(N-q)!}$$

$N=12$

Sum across a row $=0$ except for I_{12}

Sum down a column $=$ corresponding Tschebysheff coefficient

Note: When q is odd, the column is negative.

The currents are now given as a function of the parameter, z_0. z_0 may be calculated from $T_{2N-1}(z_0)=r$ using (11). However, it will now be shown that z_0 may be computed from a much simpler formula. Heretofore, $T_n(z)$ was given by cos (n arccos z) in the region $|z| \leqq 1$, and it was noted that $T_n(z)$ could not exceed 1 there. When the limits of z were extended to $\pm\infty$, the polynomial form or the closed form (11) had to be employed, particularly when solving the equation $T_{2N-1}(z_0)=r$ for $r>1$. However, it is possible to modify the simple cosine form of $T_n(z)$ so that it will apply to the region $|z| \geqq 1$.

From (5), $T_n(z)=\cos(n \arccos z)=r$, but here $r>1$. For the cosine of an argument to exceed one, the argument must be imaginary. Therefore, let $\cos inu=r$ and $iu=\arccos z$. The above is equivalent to setting $\cosh nu=r$ and $u=\operatorname{arccosh} z$. Thus, for values of $|z| \geqq 1$, $T_n(z)$ may be written $T_n(z)=\cosh(n \operatorname{arccosh} z)$. The equation $\cosh[(2N-1)\operatorname{arccosh} z_0]=r$ is simply solved for z_0 to give

$$z_0 = \cosh\left(\frac{\operatorname{arccosh} r}{2N-1}\right), \tag{21}$$

whence z_0 may readily be obtained from the mathematical tables.

As further proof of the above, it will be shown that the form cosh (n arccosh z) leads to the same polynomial in z as does cos (n arccos z). Whence it can be concluded that the cosh formula represents the Tschebyscheff polynomial, but in a different region. It was noted above that $\cos n\theta$ may be expanded into a polynomial in $x=\cos\theta$. This fact may be derived from $(\cos n\theta+i \sin n\theta) =(\cos\theta+i\sin\theta)^n$. Expanding the terms on the right and equating real and imaginary terms yields

$$\cos n\theta = \cos^n\theta - \binom{n}{2}\cos^{n-2}\theta\sin^2\theta + \binom{n}{4}\cos^{n-4}\theta\sin^4\theta\cdots, \tag{22}$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

If $\sin^2\theta$ is replaced by $(1-\cos^2\theta)$, it is apparent that $\cos n\theta$ will emerge as a polynomial in powers of $\cos\theta$.

The expansion of $\cosh n\theta$ may be effected in a somewhat similar manner. Since $\cosh n\theta+\sinh n\theta=e^{n\theta} =(e^\theta)^n=(\cosh\theta+\sinh\theta)^n$ and $\cosh n\theta-\sinh n\theta=e^{-n\theta} =(e^{-\theta})^n=(\cosh\theta-\sinh\theta)^n$, we may solve for $\cosh n\theta$ and expand so that

$$\cosh n\theta = \cosh^{n\theta} + \binom{n}{2}\cosh^{n-2}\theta\sinh^2\theta + \binom{n}{4}\cos^{n-4}\theta\sinh^4\theta\cdots. \tag{23}$$

If $\sinh^2\theta$ is replaced by $-(1-\cosh^2\theta)$, (23) will have exactly the same coefficients as (22).

As an afterthought, it may be of interest to note that the above might be more simply established by reasoning that if $\cos n\theta=P(\cos\theta)$ where P denotes some polynomial one may also write $\cos in\theta=P(\cos i\theta)$ whence $\cosh n\theta=P(\cosh\theta)$.

It is now established that $\cosh n\theta$ leads to exactly the same polynomial in powers of $\cosh\theta$ as does $\cos n\theta$ in powers of $\cos\theta$. From this it follows that cosh(n arccosh z) may be expanded into a polynomial in cosh (arccosh z) $=z$ and will yield the same Tschebyscheff polynomial as the expansion of cos (n arccos z). Thus, the two formulas represent the same polynomial, except that the cosine form is applicable in the region $z\leqq 1$ and the cosh formula is applicable in the region $z\geqq 1$. The complete Tschebyscheff polynomial may now be written

$$\begin{aligned} T_n(z) &= \cos(n \arccos z); & z &\leqq 1 \\ T_n(z) &= \cosh(n \operatorname{arccosh} z); & z &\geqq 1. \end{aligned} \tag{24}$$

As a check, the equation $T_{23}(z_0)=r$ for the 24-element case was solved for several values of r by formulas (11) and (21). The z_0's resulting from the two methods were exactly the same.

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CORRECTION

N. W. Mather, author of the paper, "An Analysis of Triple-Tuned Coupled Circuits," which appeared on pages 813–822 of the July, 1950 issue of the PROCEEDINGS OF THE I.R.E., has brought the following error to the attention of the editors:

The contour values given in Fig. 8 are incorrect. The values indicated should all be halved and should agree with the tabulation in Table I.