Appendix B

Useful identities

Algebraic identities for vectors and dyadics

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \tag{B.1}$$

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \tag{B.2}$$

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \tag{B.3}$$

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \tag{B.4}$$

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \tag{B.5}$$

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) \tag{B.6}$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) = \mathbf{B} \times (\mathbf{A} \times \mathbf{C}) + \mathbf{C} \times (\mathbf{B} \times \mathbf{A}) \tag{B.7}$$

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = \mathbf{A} \cdot [\mathbf{B} \times (\mathbf{C} \times \mathbf{D})] = (\mathbf{B} \cdot \mathbf{D})(\mathbf{A} \cdot \mathbf{C}) - (\mathbf{B} \cdot \mathbf{C})(\mathbf{A} \cdot \mathbf{D}) \tag{B.8}$$

$$(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = \mathbf{C}[\mathbf{A} \cdot (\mathbf{B} \times \mathbf{D})] - \mathbf{D}[\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})] \tag{B.9}$$

$$\mathbf{A} \times [\mathbf{B} \times (\mathbf{C} \times \mathbf{D})] = (\mathbf{B} \cdot \mathbf{D})(\mathbf{A} \times \mathbf{C}) - (\mathbf{B} \cdot \mathbf{C})(\mathbf{A} \times \mathbf{D}) \tag{B.10}$$

$$\mathbf{A} \cdot (\bar{\mathbf{c}} \cdot \mathbf{B}) = (\mathbf{A} \cdot \bar{\mathbf{c}}) \cdot \mathbf{B} \tag{B.11}$$

$$\mathbf{A} \times (\bar{\mathbf{c}} \times \mathbf{B}) = (\mathbf{A} \times \bar{\mathbf{c}}) \times \mathbf{B} \tag{B.12}$$

$$\mathbf{C} \cdot (\bar{\mathbf{a}} \cdot \bar{\mathbf{b}}) = (\mathbf{C} \cdot \bar{\mathbf{a}}) \cdot \bar{\mathbf{b}} \tag{B.13}$$

$$(\bar{\mathbf{a}} \cdot \bar{\mathbf{b}}) \cdot \mathbf{C} = \bar{\mathbf{a}} \cdot (\bar{\mathbf{b}} \cdot \mathbf{C}) \tag{B.14}$$

$$\mathbf{A} \cdot (\mathbf{B} \times \bar{\mathbf{c}}) = -\mathbf{B} \cdot (\mathbf{A} \times \bar{\mathbf{c}}) = (\mathbf{A} \times \mathbf{B}) \cdot \bar{\mathbf{c}} \tag{B.15}$$

$$\mathbf{A} \times (\mathbf{B} \times \bar{\mathbf{c}}) = \mathbf{B} \cdot (\mathbf{A} \times \bar{\mathbf{c}}) - \bar{\mathbf{c}}(\mathbf{A} \cdot \mathbf{B}) \tag{B.16}$$

$$\mathbf{A} \cdot \bar{\mathbf{I}} = \bar{\mathbf{I}} \cdot \mathbf{A} = \mathbf{A} \tag{B.17}$$

Integral theorems

Note: S bounds V, Γ bounds S, $\hat{\mathbf{n}}$ is normal to S at $\mathbf{r}$, $\hat{\mathbf{l}}$ and $\hat{\mathbf{m}}$ are tangential to S at $\mathbf{r}$, $\hat{\mathbf{l}}$ is tangential to the contour Γ, $\hat{\mathbf{m}} \times \hat{\mathbf{l}} = \hat{\mathbf{n}}$, $\mathbf{dl} = \hat{\mathbf{l}}\, dl$, and $\mathbf{dS} = \hat{\mathbf{n}}\, dS$.

Divergence theorem

$$\int_V \nabla \cdot \mathbf{A}\, dV = \oint_S \mathbf{A} \cdot \mathbf{dS} \tag{B.18}$$

$$\int_V \nabla \cdot \bar{\mathbf{a}}\, dV = \oint_S \hat{\mathbf{n}} \cdot \bar{\mathbf{a}}\, dS \tag{B.19}$$

$$\int_S \nabla_s \cdot \mathbf{A}\, dS = \oint_\Gamma \hat{\mathbf{m}} \cdot \mathbf{A}\, dl \tag{B.20}$$

Gradient theorem

$$\int_V \nabla a\, dV = \oint_S a\mathbf{dS} \tag{B.21}$$

$$\int_V \nabla \mathbf{A}\, dV = \oint_S \hat{\mathbf{n}}\mathbf{A}\, dS \tag{B.22}$$

$$\int_V \nabla_s a\, dS = \oint_\Gamma \hat{\mathbf{m}} a\, dl \tag{B.23}$$

Curl theorem

$$\int_V (\nabla \times \mathbf{A})\, dV = -\oint_S \mathbf{A} \times \mathbf{dS} \tag{B.24}$$

$$\int_V (\nabla \times \bar{\mathbf{a}})\, dV = \oint_S \hat{\mathbf{n}} \times \bar{\mathbf{a}}\, dS \tag{B.25}$$

$$\int_S \nabla_s \times \mathbf{A}\, dS = \oint_\Gamma \hat{\mathbf{m}} \times \mathbf{A}\, dl \tag{B.26}$$

Stokes's theorem

$$\int_S (\nabla \times \mathbf{A}) \cdot \mathbf{dS} = \oint_\Gamma \mathbf{A} \cdot \mathbf{dl} \tag{B.27}$$

$$\int_S \hat{\mathbf{n}} \cdot (\nabla \times \bar{\mathbf{a}})\, dS = \oint_\Gamma \mathbf{dl} \cdot \bar{\mathbf{a}} \tag{B.28}$$

Green's first identity for scalar fields

$$\int_V (\nabla a \cdot \nabla b + a\nabla^2 b)\, dV = \oint_S a \frac{\partial b}{\partial n}\, dS \tag{B.29}$$

Green's second identity for scalar fields (Green's theorem)

$$\int_V (a\nabla^2 b - b\nabla^2 a)\, dV = \oint_S \left(a\frac{\partial b}{\partial n} - b\frac{\partial a}{\partial n} \right) dS \tag{B.30}$$

Green's first identity for vector fields

$$\begin{aligned} &\int_V \{(\nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{B}) - \mathbf{A} \cdot [\nabla \times (\nabla \times \mathbf{B})]\}\, dV = \\ &\qquad \int_V \nabla \cdot [\mathbf{A} \times (\nabla \times \mathbf{B})]\, dV = \oint_S [\mathbf{A} \times (\nabla \times \mathbf{B})] \cdot \mathbf{dS} \end{aligned} \tag{B.31}$$

Green's second identity for vector fields

$$\begin{aligned} &\int_V \{\mathbf{B} \cdot [\nabla \times (\nabla \times \mathbf{A})] - \mathbf{A} \cdot [\nabla \times (\nabla \times \mathbf{B})]\}\, dV = \\ &\qquad \oint_S [\mathbf{A} \times (\nabla \times \mathbf{B}) - \mathbf{B} \times (\nabla \times \mathbf{A})] \cdot \mathbf{dS} \end{aligned} \tag{B.32}$$

Helmholtz theorem

$$\mathbf{A}(\mathbf{r}) = -\nabla \left[\int_V \frac{\nabla' \cdot \mathbf{A}(\mathbf{r}')}{4\pi |\mathbf{r} - \mathbf{r}'|} \, dV' - \oint_S \frac{\mathbf{A}(\mathbf{r}') \cdot \hat{\mathbf{n}}'}{4\pi |\mathbf{r} - \mathbf{r}'|} \, dS' \right] + \\ + \nabla \times \left[\int_V \frac{\nabla' \times \mathbf{A}(\mathbf{r}')}{4\pi |\mathbf{r} - \mathbf{r}'|} \, dV' + \oint_S \frac{\mathbf{A}(\mathbf{r}') \times \hat{\mathbf{n}}'}{4\pi |\mathbf{r} - \mathbf{r}'|} \, dS' \right] \tag{B.33}$$

Miscellaneous identities

$$\oint_S \mathbf{dS} = 0 \tag{B.34}$$

$$\int_S \hat{\mathbf{n}} \times (\nabla a) \, dS = \oint_\Gamma a \mathbf{dl} \tag{B.35}$$

$$\int_S (\nabla a \times \nabla b) \cdot \mathbf{dS} = \int_\Gamma a \nabla b \cdot \mathbf{dl} = -\int_\Gamma b \nabla a \cdot \mathbf{dl} \tag{B.36}$$

$$\oint \mathbf{dl} \, \mathbf{A} = \int_S \hat{\mathbf{n}} \times (\nabla \mathbf{A}) \, dS \tag{B.37}$$

Derivative identities

$$\nabla (a + b) = \nabla a + \nabla b \tag{B.38}$$

$$\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B} \tag{B.39}$$

$$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B} \tag{B.40}$$

$$\nabla (ab) = a \nabla b + b \nabla a \tag{B.41}$$

$$\nabla \cdot (a\mathbf{B}) = a \nabla \cdot \mathbf{B} + \mathbf{B} \cdot \nabla a \tag{B.42}$$

$$\nabla \times (a\mathbf{B}) = a \nabla \times \mathbf{B} - \mathbf{B} \times \nabla a \tag{B.43}$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B} \tag{B.44}$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} \tag{B.45}$$

$$\nabla (\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} \tag{B.46}$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \tag{B.47}$$

$$\nabla \cdot (\nabla a) = \nabla^2 a \tag{B.48}$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0 \tag{B.49}$$

$$\nabla \times (\nabla a) = 0 \tag{B.50}$$

$$\nabla \times (a \nabla b) = \nabla a \times \nabla b \tag{B.51}$$

$$\nabla^2 (ab) = a \nabla^2 b + 2(\nabla a) \cdot (\nabla b) + b \nabla^2 a \tag{B.52}$$

$$\nabla^2 (a\mathbf{B}) = a \nabla^2 \mathbf{B} + \mathbf{B} \nabla^2 a + 2(\nabla a \cdot \nabla)\mathbf{B} \tag{B.53}$$

$$\nabla^2 \bar{\mathbf{a}} = \nabla (\nabla \cdot \bar{\mathbf{a}}) - \nabla \times (\nabla \times \bar{\mathbf{a}}) \tag{B.54}$$

$$\nabla \cdot (\mathbf{AB}) = (\nabla \cdot \mathbf{A})\mathbf{B} + \mathbf{A} \cdot (\nabla \mathbf{B}) = (\nabla \cdot \mathbf{A})\mathbf{B} + (\mathbf{A} \cdot \nabla)\mathbf{B} \tag{B.55}$$

$$\nabla \times (\mathbf{AB}) = (\nabla \times \mathbf{A})\mathbf{B} - \mathbf{A} \times (\nabla \mathbf{B}) \tag{B.56}$$

$$\nabla \cdot (\nabla \times \bar{\mathbf{a}}) = 0 \tag{B.57}$$

$$\nabla \times (\nabla \mathbf{A}) = 0 \tag{B.58}$$

$$\nabla(\mathbf{A} \times \mathbf{B}) = (\nabla \mathbf{A}) \times \mathbf{B} - (\nabla \mathbf{B}) \times \mathbf{A} \tag{B.59}$$

$$\nabla(a\mathbf{B}) = (\nabla a)\mathbf{B} + a(\nabla \mathbf{B}) \tag{B.60}$$

$$\nabla \cdot (a\bar{\mathbf{b}}) = (\nabla a) \cdot \bar{\mathbf{b}} + a(\nabla \cdot \bar{\mathbf{b}}) \tag{B.61}$$

$$\nabla \times (a\bar{\mathbf{b}}) = (\nabla a) \times \bar{\mathbf{b}} + a(\nabla \times \bar{\mathbf{b}}) \tag{B.62}$$

$$\nabla \cdot (a\bar{\mathbf{I}}) = \nabla a \tag{B.63}$$

$$\nabla \times (a\bar{\mathbf{I}}) = \nabla a \times \bar{\mathbf{I}} \tag{B.64}$$

Identities involving the displacement vector

Note: $\mathbf{R} = \mathbf{r} - \mathbf{r}'$, $R = |\mathbf{R}|$, $\hat{\mathbf{R}} = \mathbf{R}/R$, $f'(x) = df(x)/dx$.

$$\nabla f(R) = -\nabla' f(R) = \hat{\mathbf{R}} f'(R) \tag{B.65}$$

$$\nabla R = \hat{\mathbf{R}} \tag{B.66}$$

$$\nabla \left(\frac{1}{R}\right) = -\frac{\hat{\mathbf{R}}}{R^2} \tag{B.67}$$

$$\nabla \left(\frac{e^{-jkR}}{R}\right) = -\hat{\mathbf{R}} \left(\frac{1}{R} + jk\right) \frac{e^{-jkR}}{R} \tag{B.68}$$

$$\nabla \cdot \left[f(R)\hat{\mathbf{R}}\right] = -\nabla' \cdot \left[f(R)\hat{\mathbf{R}}\right] = 2\frac{f(R)}{R} + f'(R) \tag{B.69}$$

$$\nabla \cdot \mathbf{R} = 3 \tag{B.70}$$

$$\nabla \cdot \hat{\mathbf{R}} = \frac{2}{R} \tag{B.71}$$

$$\nabla \cdot \left(\hat{\mathbf{R}} \frac{e^{-jkR}}{R}\right) = \left(\frac{1}{R} - jk\right) \frac{e^{-jkR}}{R} \tag{B.72}$$

$$\nabla \times \left[f(R)\hat{\mathbf{R}}\right] = 0 \tag{B.73}$$

$$\nabla^2 \left(\frac{1}{R}\right) = -4\pi \delta(\mathbf{R}) \tag{B.74}$$

$$(\nabla^2 + k^2) \frac{e^{-jkR}}{R} = -4\pi \delta(\mathbf{R}) \tag{B.75}$$

Identities involving the plane-wave function

Note: $\mathbf{E}$ is a constant vector, $k = |\mathbf{k}|$.

$$\nabla \left(e^{-j\mathbf{k}\cdot\mathbf{r}}\right) = -j\mathbf{k}e^{-j\mathbf{k}\cdot\mathbf{r}} \tag{B.76}$$

$$\nabla \cdot \left(\mathbf{E}e^{-j\mathbf{k}\cdot\mathbf{r}}\right) = -j\mathbf{k} \cdot \mathbf{E}e^{-j\mathbf{k}\cdot\mathbf{r}} \tag{B.77}$$

$$\nabla \times \left(\mathbf{E}e^{-j\mathbf{k}\cdot\mathbf{r}}\right) = -j\mathbf{k} \times \mathbf{E}e^{-j\mathbf{k}\cdot\mathbf{r}} \tag{B.78}$$

$$\nabla^2 \left(\mathbf{E}e^{-j\mathbf{k}\cdot\mathbf{r}}\right) = -k^2\mathbf{E}e^{-j\mathbf{k}\cdot\mathbf{r}} \tag{B.79}$$

Identities involving the transverse/longitudinal decomposition

Note: $\hat{\mathbf{u}}$ is a constant unit vector, $A_u \equiv \hat{\mathbf{u}} \cdot \mathbf{A}$, $\partial/\partial u \equiv \hat{\mathbf{u}} \cdot \nabla$, $\mathbf{A}_t \equiv \mathbf{A} - \hat{\mathbf{u}}A_u$, $\nabla_t \equiv \nabla - \hat{\mathbf{u}}\partial/\partial u$.

$$\mathbf{A} = \mathbf{A}_t + \hat{\mathbf{u}}A_u \tag{B.80}$$

$$\nabla = \nabla_t + \hat{\mathbf{u}}\frac{\partial}{\partial u} \tag{B.81}$$

$$\hat{\mathbf{u}} \cdot \mathbf{A}_t = 0 \tag{B.82}$$

$$(\hat{\mathbf{u}} \cdot \nabla_t)\,\phi = 0 \tag{B.83}$$

$$\nabla_t\phi = \nabla\phi - \hat{\mathbf{u}}\frac{\partial\phi}{\partial u} \tag{B.84}$$

$$\hat{\mathbf{u}} \cdot (\nabla\phi) = (\hat{\mathbf{u}} \cdot \nabla)\phi = \frac{\partial\phi}{\partial u} \tag{B.85}$$

$$\hat{\mathbf{u}} \cdot (\nabla_t\phi) = 0 \tag{B.86}$$

$$\nabla_t \cdot (\hat{\mathbf{u}}\phi) = 0 \tag{B.87}$$

$$\nabla_t \times (\hat{\mathbf{u}}\phi) = -\hat{\mathbf{u}} \times \nabla_t\phi \tag{B.88}$$

$$\nabla_t \times (\hat{\mathbf{u}} \times \mathbf{A}) = \hat{\mathbf{u}}\nabla_t \cdot \mathbf{A}_t \tag{B.89}$$

$$\hat{\mathbf{u}} \times (\nabla_t \times \mathbf{A}) = \nabla_t A_u \tag{B.90}$$

$$\hat{\mathbf{u}} \times (\nabla_t \times \mathbf{A}_t) = 0 \tag{B.91}$$

$$\hat{\mathbf{u}} \cdot (\hat{\mathbf{u}} \times \mathbf{A}) = 0 \tag{B.92}$$

$$\hat{\mathbf{u}} \times (\hat{\mathbf{u}} \times \mathbf{A}) = -\mathbf{A}_t \tag{B.93}$$

$$\nabla\phi = \nabla_t\phi + \hat{\mathbf{u}}\frac{\partial\phi}{\partial u} \tag{B.94}$$

$$\nabla \cdot \mathbf{A} = \nabla_t \cdot \mathbf{A}_t + \frac{\partial A_u}{\partial u} \tag{B.95}$$

$$\nabla \times \mathbf{A} = \nabla_t \times \mathbf{A}_t + \hat{\mathbf{u}} \times \left[\frac{\partial\mathbf{A}_t}{\partial u} - \nabla_t A_u\right] \tag{B.96}$$

$$\nabla^2\phi = \nabla_t^2\phi + \frac{\partial^2\phi}{\partial u^2} \tag{B.97}$$

$$\nabla \times \nabla \times \mathbf{A} = \left[\nabla_t \times \nabla_t \times \mathbf{A}_t - \frac{\partial^2\mathbf{A}_t}{\partial u^2} + \nabla_t\frac{\partial A_u}{\partial u}\right] + \hat{\mathbf{u}}\left[\frac{\partial}{\partial u}(\nabla_t \cdot \mathbf{A}_t) - \nabla_t^2 A_u\right] \tag{B.98}$$

$$\nabla^2\mathbf{A} = \left[\nabla_t(\nabla_t \cdot \mathbf{A}_t) + \frac{\partial^2\mathbf{A}_t}{\partial u^2} - \nabla_t \times \nabla_t \times \mathbf{A}_t\right] + \hat{\mathbf{u}}\nabla^2 A_u \tag{B.99}$$