Appendix 4

Dyadic Analysis*

DEFINITIONS

Vector $\mathbf{d}'$ is a linear vector function of vector $\mathbf{d}$ when the following relationships hold:

$$d'_x = a_{xx}d_x + a_{xy}d_y + a_{xz}d_z$$
$$d'_y = a_{yx}d_x + a_{yy}d_y + a_{yz}d_z$$
$$d'_z = a_{zx}d_x + a_{zy}d_y + a_{zz}d_z. \tag{A4.1}$$

These relationships can be represented in more compact form by means of the matrix notation

$$\mathbf{d}' = \overline{\overline{a}}\cdot\mathbf{d}. \tag{A4.2}$$

The matrix operator itself can be expressed in terms of dyads as

$$\overline{\overline{a}} = a_{xx}\mathbf{u}_x\mathbf{u}_x + a_{xy}\mathbf{u}_x\mathbf{u}_y + a_{xz}\mathbf{u}_x\mathbf{u}_z + a_{yx}\mathbf{u}_y\mathbf{u}_x + a_{yy}\mathbf{u}_y\mathbf{u}_y + a_{yz}\mathbf{u}_y\mathbf{u}_z + a_{zx}\mathbf{u}_z\mathbf{u}_x + a_{zy}\mathbf{u}_z\mathbf{u}_y + a_{zz}\mathbf{u}_z\mathbf{u}_z \tag{A4.3}$$

provided, by convention, $\mathbf{ab}\cdot\mathbf{c}$ stands for $\mathbf{a}(\mathbf{b}\cdot\mathbf{c})$. The symbol $\mathbf{ab}$ is called a *dyad*, and a sum of dyads such as $\overline{\overline{a}}$ is a *dyadic*. Also by convention, $\mathbf{c}\cdot\mathbf{ab}$ stands for $(\mathbf{c}\cdot\mathbf{a})\mathbf{b}$, so that the dot product of a dyad and a vector is now defined for $\mathbf{ab}$ acting as both a prefactor and a postfactor. The writing of $\overline{\overline{a}}$ in "nonion" form, as shown above, is rather cumbersome, and one often prefers to use the form

$$\overline{\overline{a}} = (a_{xx}\mathbf{u}_x + a_{yx}\mathbf{u}_y + a_{zx}\mathbf{u}_z)\mathbf{u}_x + (a_{xy}\mathbf{u}_x + a_{yy}\mathbf{u}_y + a_{zy}\mathbf{u}_z)\mathbf{u}_y + (a_{xz}\mathbf{u}_x + a_{yz}\mathbf{u}_y + a_{zz}\mathbf{u}_z)\mathbf{u}_z = \mathbf{a}'_x\mathbf{u}_x + \mathbf{a}'_y\mathbf{u}_y + \mathbf{a}'_z\mathbf{u}_z \tag{A4.4}$$

where the $\mathbf{a}'$ are the column vectors of the matrix of $\overline{\overline{a}}$. Alternatively,

$$\overline{\overline{a}} = \mathbf{u}_x(a_{xx}\mathbf{u}_x + a_{xy}\mathbf{u}_y + a_{xz}\mathbf{u}_z) + \mathbf{u}_y(a_{yx}\mathbf{u}_x + a_{yy}\mathbf{u}_y + a_{yz}\mathbf{u}_z) + \mathbf{u}_z(a_{zx}\mathbf{u}_x + a_{zy}\mathbf{u}_y + a_{zz}\mathbf{u}_z) = \mathbf{u}_x\mathbf{a}_x + \mathbf{u}_y\mathbf{a}_y + \mathbf{u}_z\mathbf{a}_z, \tag{A4.5}$$

*Professor Lindell has been kind enough to check this appendix, make corrections, and suggest additional formulas.

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where the **a** are the row vectors of the matrix of $\bar{\bar{a}}$. It is obvious that $\bar{\bar{a}}\cdot\mathbf{d}$ is, in general, different from $\mathbf{d}\cdot\bar{\bar{a}}$. In other words, the order in which $\bar{\bar{a}}$ and **d** appear should be carefully respected. $\bar{\bar{a}}\cdot\mathbf{d}$ is equal to $\mathbf{d}\cdot\bar{\bar{a}}$ only when the dyadic is symmetric (i.e., when $a_{ik} = a_{ki}$). The *transpose* of $\bar{\bar{a}}$ is *a dyadic* $\bar{\bar{a}}^t$ such that $\bar{\bar{a}}\cdot\mathbf{d}$ is equal to $\mathbf{d}\cdot\bar{\bar{a}}^t$. One may easily check that the transpose is obtained by an interchange of rows and columns. More precisely,

$$\bar{\bar{a}}^t = \mathbf{a}_x\mathbf{u}_x + \mathbf{a}_y\mathbf{u}_y + \mathbf{a}_z\mathbf{u}_z = \mathbf{u}_x\mathbf{a}'_x + \mathbf{u}_y\mathbf{a}'_y + \mathbf{u}_z\mathbf{a}'_z. \quad \text{(A4.6)}$$

The *trace of the dyadic* is the sum of its diagonal terms. Thus,

$$tr\,\bar{\bar{a}} = a_{xx} + a_{yy} + a_{zz}. \quad \text{(A4.7)}$$

The trace is a scalar (i.e., it is invariant with respect to orthogonal transformations of the base vectors). The trace of **ab** is $\mathbf{a}\cdot\mathbf{b}$. Among dyadics endowed with special properties we note

1. The *unitary dyadic*, which represents a pure rotation. The determinant of its elements is equal to 1.
2. The *identity dyadic*

$$\bar{\bar{I}} = \mathbf{u}_x\mathbf{u}_x + \mathbf{u}_y\mathbf{u}_y + \mathbf{u}_z\mathbf{u}_z. \quad \text{(A4.8)}$$

Clearly,

$$\bar{\bar{I}}\cdot\mathbf{d} = \mathbf{d}\cdot\bar{\bar{I}} = \mathbf{d}. \quad \text{(A4.9)}$$

3. The *symmetric dyadic*, characterized by $a_{ik} = a_{ki}$, for which $\bar{\bar{a}}^t = \bar{\bar{a}}$. The dyadic **ab** is symmetric when $\mathbf{a} \times \mathbf{b} = 0$. Further,

$$\bar{\bar{a}}\cdot\mathbf{d} = \mathbf{d}\cdot\bar{\bar{a}}. \quad \text{(A4.10)}$$

4. The *antisymmetric dyadic*, characterized by $a_{ik} = -a_{ki}$. For such a dyadic $\bar{\bar{a}}^t = -\bar{\bar{a}}$, and

$$\bar{\bar{a}}\cdot\mathbf{d} = -\mathbf{d}\cdot\bar{\bar{a}}. \quad \text{(A4.11)}$$

The diagonal elements are zero, and there are only three distinct components. The dyadic can always be written in terms of $\bar{\bar{I}}$ and a suitable vector **b** as

$$\begin{aligned}\bar{\bar{a}} &= -b_z\mathbf{u}_x\mathbf{u}_y + b_y\mathbf{u}_x\mathbf{u}_z + b_z\mathbf{u}_y\mathbf{u}_x \\ &\quad - b_x\mathbf{u}_y\mathbf{u}_z - b_y\mathbf{u}_z\mathbf{u}_x + b_x\mathbf{u}_z\mathbf{u}_y, \\ &= \bar{\bar{I}} \times \mathbf{b},\end{aligned} \quad \text{(A4.12)}$$

where the *skew product* is the dyad

$$(\mathbf{bc}) \times \mathbf{d} = \mathbf{b}(\mathbf{c} \times \mathbf{d}). \quad \text{(A4.13)}$$

The antisymmetric $\bar{\bar{a}}$ can also be expressed as

$$\bar{\bar{a}} = \mathbf{cb} - \mathbf{bc}. \quad \text{(A4.14)}$$

5. The *reflection dyadic*

$$\overline{\overline{r}}_f(\mathbf{u}) = \overline{\overline{I}} - 2\mathbf{uu}, \tag{A4.15}$$

where $\mathbf{u}$ is a (real) unit vector. Applied to the position vector $\mathbf{r}$, it performs a reflection with respect to a plane perpendicular to $\mathbf{u}$.

6. The *rotation dyadic*

$$\overline{\overline{r}}_r(\mathbf{u}) = \mathbf{uu} + \sin\theta(\mathbf{u} \times \overline{\overline{I}}) + \cos\theta(\overline{\overline{I}} - \mathbf{uu}). \tag{A4.16}$$

Applied to a vector, it performs a rotation by an angle θ in the right-hand direction around the direction of $\mathbf{u}$.

The elements of a dyadic may be complex (a case in point is the free-space dyadic discussed in Chapter 7). It then becomes useful to introduce concepts such as the *Hermitian dyadic* ($a_{ik} = a_{ki}^*$), or the *anti-Hermitian dyadic* ($a_{ik} = -a_{ki}^*$). Useful *products of dyads* are defined as follows:

$$(\mathbf{ab})\cdot(\mathbf{cd}) = \mathbf{a}(\mathbf{b}\cdot\mathbf{c})\mathbf{d} \quad \text{(the direct product, a dyad).} \tag{A4.17}$$

$$(\mathbf{ab}) : (\mathbf{cd}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) \quad \text{(the double product, a scalar).} \tag{A4.18}$$

$$(\mathbf{ab}) \overset{\times}{\times} (\mathbf{cd}) = (\mathbf{a} \times \mathbf{c})(\mathbf{b} \times \mathbf{d}) \quad \text{(the double cross-product, a dyad).} \tag{A4.19}$$

$$(\mathbf{ab}) \overset{\times}{\cdot} (\mathbf{cd}) = (\mathbf{a} \times \mathbf{c})(\mathbf{b}\cdot\mathbf{d}) \quad \text{(a vector).} \tag{A4.20}$$

$$(\mathbf{ab}) \overset{\cdot}{\times} (\mathbf{cd}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b} \times \mathbf{d}) \quad \text{(a vector).} \tag{A4.21}$$

General Multiplicative Relationships

$$(\mathbf{b}\cdot\overline{\overline{a}})\cdot\mathbf{c} = \mathbf{b}\cdot(\overline{\overline{a}}\cdot\mathbf{c}) = \mathbf{b}\cdot\overline{\overline{a}}\cdot\mathbf{c} \tag{A4.22}$$

$$(\mathbf{b} \times \mathbf{c})\cdot\overline{\overline{a}} = \mathbf{b}\cdot(\mathbf{c} \times \overline{\overline{a}}) = -\mathbf{c}\cdot(\mathbf{b} \times \overline{\overline{a}}) \tag{A4.23}$$

$$(\overline{\overline{a}} \times \mathbf{b})\cdot\mathbf{c} = \overline{\overline{a}}\cdot(\mathbf{b} \times \mathbf{c}) = -(\overline{\overline{a}} \times \mathbf{c})\cdot\mathbf{b} \quad \text{(but not } (\overline{\overline{a}}\cdot\mathbf{b}) \times \mathbf{c}) \tag{A4.24}$$

$$(\mathbf{b} \times \overline{\overline{a}})\cdot\mathbf{c} = \mathbf{b} \times (\overline{\overline{a}}\cdot\mathbf{b}) \tag{A4.25}$$

$$(\mathbf{b}\cdot\overline{\overline{a}}) \times \mathbf{c} = \mathbf{b}\cdot(\overline{\overline{a}} \times \mathbf{c}) \tag{A4.26}$$

$$(\mathbf{b} \times \overline{\overline{a}}) \times \mathbf{c} = \mathbf{b} \times (\overline{\overline{a}} \times \mathbf{c}) = \mathbf{b} \times \overline{\overline{a}} \times \mathbf{c} \tag{A4.27}$$

$$\mathbf{b} \times (\mathbf{c} \times \overline{\overline{a}}) = \mathbf{c}(\mathbf{b}\cdot\overline{\overline{a}}) - \overline{\overline{a}}(\mathbf{b}\cdot\mathbf{c}) \tag{A4.28}$$

$$(\mathbf{bc} - \mathbf{cb})\cdot\mathbf{d} = (\mathbf{c} \times \mathbf{b}) \times \mathbf{d} \tag{A4.29}$$

$$(\mathbf{c}\cdot\overline{\overline{a}})\cdot\overline{\overline{b}} = \mathbf{c}\cdot(\overline{\overline{a}}\cdot\overline{\overline{b}}) = \mathbf{c}\cdot\overline{\overline{a}}\cdot\overline{\overline{b}} \tag{A4.30}$$

$$(\overline{\overline{a}}\cdot\overline{\overline{b}})\cdot\mathbf{c} = \overline{\overline{a}}\cdot(\overline{\overline{b}}\cdot\mathbf{c}) = \overline{\overline{a}}\cdot\overline{\overline{b}}\cdot\mathbf{c} \tag{A4.31}$$

$$(\mathbf{c} \times \overline{\overline{a}})\cdot\overline{\overline{b}} = \mathbf{c} \times (\overline{\overline{a}}\cdot\overline{\overline{b}}) = \mathbf{c} \times \overline{\overline{a}}\cdot\mathbf{b} \tag{A4.32}$$

$$(\overline{\overline{a}}\cdot\overline{\overline{b}}) \times \mathbf{c} = \overline{\overline{a}}\cdot(\overline{\overline{b}} \times \mathbf{c}) = \overline{\overline{a}}\cdot\overline{\overline{b}} \times \mathbf{c} \tag{A4.33}$$

$$(\overline{\overline{a}} \times \mathbf{c})\cdot\overline{\overline{b}} = \overline{\overline{a}}\cdot(\mathbf{c} \times \overline{\overline{b}}) \tag{A4.34}$$

$$\mathbf{b}\cdot\overline{\overline{a}}\cdot\mathbf{c} = \mathbf{c}\cdot\overline{\overline{a}}^t\cdot\mathbf{b} \tag{A4.35}$$

$$\overline{\overline{a}}\cdot(\overline{\overline{b}}\cdot\overline{\overline{c}}) = (\overline{\overline{a}}\cdot\overline{\overline{b}})\cdot\overline{\overline{c}}. \tag{A4.36}$$

The identity dyadic satisfies the following relationships:

$$(\overline{\overline{I}} \times \mathbf{b})\cdot\mathbf{c} = \mathbf{b}\cdot(\overline{\overline{I}} \times \mathbf{c}) = \mathbf{b} \times \mathbf{c} \tag{A4.37}$$

$$(\overline{\overline{I}} \times \mathbf{b})\cdot\overline{\overline{a}} = \mathbf{b} \times \overline{\overline{a}} = (\mathbf{b} \times \overline{\overline{I}})\cdot\overline{\overline{a}} \tag{A4.38}$$

$$\overline{\overline{I}} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{cb} - \mathbf{bc}. \tag{A4.39}$$

DIFFERENTIAL RELATIONSHIPS

Differentiation with Respect to a Parameter

$$\frac{d}{dt}(f\overline{\overline{a}}) = \frac{df}{dt}\overline{\overline{a}} + f\frac{d\,\overline{\overline{a}}}{dt} \tag{A4.40}$$

$$\frac{d}{dt}(\overline{\overline{a}}\cdot\mathbf{b}) = \frac{d\,\overline{\overline{a}}}{dt}\cdot\mathbf{b} + \overline{\overline{a}}\cdot\frac{d\mathbf{b}}{dt} \tag{A4.41}$$

$$\frac{d}{dt}(\overline{\overline{a}} \times \mathbf{b}) = \frac{d\,\overline{\overline{a}}}{dt} \times \mathbf{b} + \overline{\overline{a}} \times \frac{d\mathbf{b}}{dt} \tag{A4.42}$$

$$\frac{d}{dt}(\overline{\overline{a}}\cdot\overline{\overline{b}}) = \frac{d\,\overline{\overline{a}}}{dt}\cdot\overline{\overline{b}} + \overline{\overline{a}}\cdot\frac{d\overline{\overline{b}}}{dt}. \tag{A4.43}$$

Basic Differential Operators

The action of a linear operator $\mathcal{L}$ on a dyadic is defined by the formula

$$\mathcal{L}\overline{\overline{a}} = (\mathcal{L}\mathbf{a}'_x)\mathbf{u}_x + (\mathcal{L}\mathbf{a}'_y)\mathbf{u}_y + (\mathcal{L}\mathbf{a}'_z)\mathbf{u}_z. \tag{A4.44}$$

In particular,

$$\begin{aligned}\operatorname{div}\overline{\overline{a}} = \nabla\cdot\overline{\overline{a}} &= (\operatorname{div}\mathbf{a}'_x)\mathbf{u}_x + (\operatorname{div}\mathbf{a}'_y)\mathbf{u}_y + (\operatorname{div}\mathbf{a}'_z)\mathbf{u}_z \\ &= \frac{\partial\mathbf{a}_x}{\partial x} + \frac{\partial\mathbf{a}_y}{\partial y} + \frac{\partial\mathbf{a}_z}{\partial z}\end{aligned} \tag{A4.45}$$

$$\begin{aligned}\operatorname{curl}\overline{\overline{a}} = \nabla \times \overline{\overline{a}} &= (\operatorname{curl}\mathbf{a}'_x)\mathbf{u}_x + (\operatorname{curl}\mathbf{a}'_y)\mathbf{u}_y + (\operatorname{curl}\mathbf{a}'_z)\mathbf{u}_z \\ &= \mathbf{u}_x\left(\frac{\partial\mathbf{a}_z}{\partial y} - \frac{\partial\mathbf{a}_y}{\partial z}\right) + \mathbf{u}_y\left(\frac{\partial\mathbf{a}_x}{\partial z} - \frac{\partial\mathbf{a}_z}{\partial x}\right) + \mathbf{u}_z\left(\frac{\partial\mathbf{a}_y}{\partial x} - \frac{\partial\mathbf{a}_x}{\partial y}\right)\end{aligned} \tag{A4.46}$$

$$\nabla^2\overline{\overline{a}} = \frac{\partial^2\overline{\overline{a}}}{\partial x^2} + \frac{\partial^2\overline{\overline{a}}}{\partial y^2} + \frac{\partial^2\overline{\overline{a}}}{\partial z^2} = \operatorname{grad}\operatorname{div}\overline{\overline{a}} - \operatorname{curl}\operatorname{curl}\overline{\overline{a}}. \tag{A4.47}$$

Also

$$\text{grad } \mathbf{a} = \nabla\mathbf{a} = \mathbf{u}_x \frac{\partial \mathbf{a}}{\partial x} + \mathbf{u}_y \frac{\partial \mathbf{a}}{\partial y} + \mathbf{u}_z \frac{\partial \mathbf{a}}{\partial z}$$

$$= \text{grad } a_x \mathbf{u}_x + \text{grad } a_y \mathbf{u}_y + \text{grad } a_z \mathbf{u}_z \quad \text{(A4.48)}$$

$$\mathbf{a} \text{ grad} = \mathbf{a}\nabla = \mathbf{a}\mathbf{u}_x \frac{\partial}{\partial x} + \mathbf{a}\mathbf{u}_y \frac{\partial}{\partial y} + \mathbf{a}\mathbf{u}_z \frac{\partial}{\partial z}. \quad \text{(A4.49)}$$

Derived Relationships

$$\text{grad}(\mathbf{b} \times \mathbf{c}) = (\text{grad } \mathbf{b}) \times \mathbf{c} - (\text{grad } \mathbf{c}) \times \mathbf{b} \quad \text{(A4.50)}$$

$$\text{grad}(f\mathbf{b}) = (\text{grad } f)\mathbf{b} + f \text{ grad } \mathbf{b} \quad (f \text{ is any scalar function}) \quad \text{(A4.51)}$$

$$(\mathbf{b}\cdot\text{grad})\overline{\overline{a}} = b_x \frac{\partial \overline{\overline{a}}}{\partial x} + b_y \frac{\partial \overline{\overline{a}}}{\partial y} + b_z \frac{\partial \overline{\overline{a}}}{\partial z} \quad \text{(A4.52)}$$

$$d\mathbf{r}\cdot\text{grad } \mathbf{a} = d\mathbf{a} \quad \text{(A4.53)}$$

$$\text{div}(\mathbf{b}\mathbf{c}) = (\text{div } \mathbf{b})\mathbf{c} + \mathbf{b}\cdot\text{grad } \mathbf{c} \quad \text{(A4.54)}$$

$$\text{div curl } \overline{\overline{a}} = 0 \quad \text{(A4.55)}$$

$$\text{div}(f\overline{\overline{a}}) = \text{grad } f\cdot\overline{\overline{a}} + f \text{ div } \overline{\overline{a}} \quad \text{(A4.56)}$$

$$\text{div}(\overline{\overline{a}}\cdot\mathbf{b}) = (\text{div } \overline{\overline{a}})\cdot\mathbf{b} + tr(\overline{\overline{a}}^t\cdot\text{grad } \mathbf{b}) \quad \text{(A4.57)}$$

$$\text{div}(\mathbf{b} \times \overline{\overline{a}}) = (\text{curl } \mathbf{b})\cdot\overline{\overline{a}} - \mathbf{b}\cdot\text{curl } \overline{\overline{a}} \quad \text{(A4.58)}$$

$$\text{div}(\mathbf{b}\mathbf{c} - \mathbf{c}\mathbf{b}) = \text{curl } (\mathbf{c} \times \mathbf{b}) \quad \text{(A4.59)}$$

$$\text{div}(f\overline{\overline{I}}) = \text{grad } f \quad \text{(A4.60)}$$

$$\text{div}(\overline{\overline{I}} \times \mathbf{a}) = \text{curl } \mathbf{a} \quad \text{(A4.61)}$$

$$\text{curl}(\mathbf{b}\mathbf{c}) = (\text{curl } \mathbf{b})\mathbf{c} - \mathbf{b} \times \text{grad } \mathbf{c} \quad \text{(A4.62)}$$

$$\text{curl grad } \mathbf{a} = 0 \quad \text{(A4.63)}$$

$$\text{curl}(f\overline{\overline{a}}) = \text{grad } f \times \overline{\overline{a}} + f \text{ curl } \overline{\overline{a}} \quad \text{(A4.64)}$$

$$\text{curl}(f\overline{\overline{I}}) = \text{grad } f \times \overline{\overline{I}} \quad \text{(A4.65)}$$

$$\text{curl}(\overline{\overline{a}} \times \mathbf{b}) = \text{curl } \overline{\overline{a}} \times \mathbf{b} - \text{grad } \mathbf{b} \overset{\times}{\times} \overline{\overline{a}} \quad \text{(A4.66)}$$

$$\text{curl curl}(f\overline{\overline{I}}) = \text{curl}(\text{grad } f \times \overline{\overline{I}}) = \text{grad grad } f - \overline{\overline{I}}\nabla^2 f. \quad \text{(A4.67)}$$

INTEGRAL RELATIONSHIPS

The integral relationships of vector analysis have their equivalent in dyadic analysis. The most important examples are

$$\int_M^N \mathbf{dc}\cdot\text{grad } \mathbf{a} = \mathbf{a}(N) - \mathbf{a}(M) \quad \text{(A4.68)}$$

$$\int_c \mathbf{dc}\ \mathbf{a} = \int_S \mathbf{u}_n \times \text{grad}\ \mathbf{a}\, dS, \tag{A4.69}$$

where the contour is described in the positive sense with respect to $\mathbf{u}_n$.

$$\int_c d\mathbf{c} \cdot \overline{\overline{a}} = \int_S \mathbf{u}_n \cdot \text{curl}\ \overline{\overline{a}}\, dS \tag{A4.70}$$

$$\int_V \text{grad}\ \mathbf{a}\, dV = \int_S \mathbf{u}_n \mathbf{a}\, dS \tag{A4.71}$$

$$\int_V \text{div}\ \overline{\overline{a}}\, dV = \int_S \mathbf{u}_n \cdot \overline{\overline{a}}\, dS \tag{A4.72}$$

$$\int_V \text{curl}\ \overline{\overline{a}}\, dV = \int_S \mathbf{u}_n \times \overline{\overline{a}}\, dS \tag{A4.73}$$

$$\int_V \left[\mathbf{b} \cdot \text{grad div}\ \overline{\overline{a}} - (\text{grad div}\ \mathbf{b}) \cdot \overline{\overline{a}}\right] dV = \int_S \left[(\mathbf{u}_n \cdot \mathbf{b})\ \text{div}\ \overline{\overline{a}} - \text{div}\ \mathbf{b}(\mathbf{u}_n \cdot \overline{\overline{a}})\right] dS \tag{A4.74}$$

$$\begin{aligned}\int_V \left[(\text{curl curl}\ \mathbf{b}) \cdot \overline{\overline{a}} - \mathbf{b} \cdot \text{curl curl}\ \overline{\overline{a}}\right] dV &= \int_S \left[(\mathbf{u}_n \times \mathbf{b}) \cdot \text{curl}\ \overline{\overline{a}} + (\mathbf{u}_n \times \text{curl}\ \mathbf{b}) \cdot \overline{\overline{a}}\right] dS \\ &= \int_S \left[\mathbf{u}_n \cdot (\mathbf{b} \times \text{curl}\ \overline{\overline{a}}) + \mathbf{u}_n \cdot (\text{curl}\ \mathbf{b} \times \overline{\overline{a}})\right] dS\end{aligned} \tag{A4.75}$$

$$\begin{aligned}\int_V \left[\mathbf{b} \cdot \nabla^2 \overline{\overline{a}} - (\nabla^2 \mathbf{b}) \cdot \overline{\overline{a}}\right] dV &= \int_S \left[(\mathbf{u}_n \cdot \mathbf{b})\ \text{div}\ \overline{\overline{a}} - \text{div}\ \mathbf{b}(\mathbf{u}_n \cdot \overline{\overline{a}}) \right. \\ &\quad \left. + \mathbf{u}_n \cdot (\mathbf{b} \times \text{curl}\ \overline{\overline{a}}) + \mathbf{u}_n \cdot (\text{curl}\ \mathbf{b} \times \overline{\overline{a}})\right] dS\end{aligned} \tag{A4.76}$$

$$\int_V (\mathbf{a} \nabla^2 f - f \nabla^2 \mathbf{a})\, dV = \int_S \mathbf{u}_n \cdot (\text{grad}\, f \mathbf{a} - f\ \text{grad}\ \mathbf{a})\, dS. \tag{A4.77}$$

RELATIONSHIPS IN CYLINDRICAL COORDINATES

Dyadic $\overline{\overline{a}}$ can be written as

$$\overline{\overline{a}} = \mathbf{a}'_r \mathbf{u}_r + \mathbf{a}'_\varphi \mathbf{u}_\varphi + \mathbf{a}'_z \mathbf{u}_z = \mathbf{u}_r \mathbf{a}_r + \mathbf{u}_\varphi \mathbf{a}_\varphi + \mathbf{u}_z \mathbf{a}_z.$$

The basic differential operators are then:

$$\begin{aligned}\text{grad}\ \mathbf{a} &= \left(\text{grad}\ a_r - \frac{a_\varphi \mathbf{u}_\varphi}{r}\right) \mathbf{u}_r + \left(\text{grad}\ a_\varphi + \frac{a_r \mathbf{u}_\varphi}{r}\right) \mathbf{u}_\varphi + \text{grad}\ a_z \mathbf{u}_z \\ &= \mathbf{u}_r \frac{\partial \mathbf{a}}{\partial r} + \mathbf{u}_\varphi \frac{1}{r} \frac{\partial \mathbf{a}}{\partial \varphi} + \mathbf{u}_z \frac{\partial \mathbf{a}}{\partial z}\end{aligned} \tag{A4.78}$$

$$\begin{aligned}\text{div}\ \overline{\overline{a}} &= \left(\text{div}\ \mathbf{a}'_r - \frac{a_{\varphi\varphi}}{r}\right) \mathbf{u}_r + \left(\text{div}\ \mathbf{a}'_\varphi + \frac{a_{\varphi r}}{r}\right) \mathbf{u}_\varphi + (\text{div}\ \mathbf{a}'_z) \mathbf{u}_z \\ &= \frac{1}{r} \mathbf{a}_r + \frac{\partial \mathbf{a}}{\partial r} + \frac{1}{r} \frac{\partial \mathbf{a}_\varphi}{\partial \varphi} + \frac{\partial \mathbf{a}_z}{\partial z}\end{aligned} \tag{A4.79}$$

$$\operatorname{curl}\overline{\overline{a}} = \left(\operatorname{curl}\mathbf{a}'_r + \frac{\mathbf{a}'_\varphi \times \mathbf{u}_\varphi}{r}\right)\mathbf{u}_r + \left(\operatorname{curl}\mathbf{a}'_\varphi - \frac{\mathbf{a}'_r \times \mathbf{u}_\varphi}{r}\right)\mathbf{u}_\varphi + \operatorname{curl}\mathbf{a}'_z\mathbf{u}_z$$

$$= \mathbf{u}_r\left(\frac{1}{r}\frac{\partial \mathbf{a}_z}{\partial \varphi} - \frac{\partial \mathbf{a}_\varphi}{\partial z}\right) + \mathbf{u}_\varphi\left(\frac{\partial \mathbf{a}_r}{\partial z} - \frac{\partial \mathbf{a}_z}{\partial r}\right) + \mathbf{u}_z\left(\frac{\mathbf{a}_\varphi}{r} + \frac{\partial \mathbf{a}_\varphi}{\partial r} - \frac{1}{r}\frac{\partial \mathbf{a}_r}{\partial \varphi}\right). \quad \text{(A4.80)}$$

In particular:

$$\operatorname{grad}\mathbf{u}_r = \frac{\mathbf{u}_\varphi\mathbf{u}_\varphi}{r} \quad \text{(A4.81)}$$

$$\operatorname{grad}\mathbf{u}_\varphi = -\frac{\mathbf{u}_\varphi\mathbf{u}_r}{r} \quad \text{(A4.82)}$$

$$\operatorname{grad}\mathbf{u}_z = 0 \quad \text{(A4.83)}$$

$$\operatorname{grad}(r\mathbf{u}_r) = \mathbf{u}_r\mathbf{u}_r + \mathbf{u}_\varphi\mathbf{u}_\varphi = \overline{\overline{I}} - \mathbf{u}_z\mathbf{u}_z. \quad \text{(A4.84)}$$

Note that the dyadic operators expressed in terms of the row vectors **a** are identical with their vector counterparts provided bars are put above scalar projections to transform them into row vectors, and provided the unit vectors are used as *prefactors*. This simple rule, which is also valid in spherical coordinates, allows one to write composite operators such as grad div simply by referring to the vector formula. For example:

$$\nabla^2\,\overline{\overline{a}} = \mathbf{u}_r\left(\nabla^2\mathbf{a}_r - \frac{\mathbf{a}_r}{r^2} - \frac{2}{r^2}\frac{\partial \mathbf{a}_\varphi}{\partial \varphi}\right) + \mathbf{u}_\varphi\left(\nabla^2\mathbf{a}_\varphi - \frac{\mathbf{a}_\varphi}{r^2} + \frac{2}{r^2}\frac{\partial \mathbf{a}_r}{\partial \varphi}\right) + \mathbf{u}_z\nabla^2\mathbf{a}_z. \quad \text{(A4.85)}$$

RELATIONSHIPS IN SPHERICAL COORDINATES

Dyadic $\overline{\overline{a}}$ can be written as

$$\overline{\overline{a}} = \mathbf{a}'_R\mathbf{u}_R + \mathbf{a}'_\theta\mathbf{u}_\theta + \mathbf{a}'_\varphi\mathbf{u}_\varphi = \mathbf{u}_R\mathbf{a}_R + \mathbf{u}_\theta\mathbf{a}_\theta + \mathbf{u}_\varphi\mathbf{a}_\varphi.$$

The basic differential operators are

$$\operatorname{grad}\mathbf{a} = \left(\operatorname{grad}a_R - \frac{a_\varphi\mathbf{u}_\varphi}{R} - \frac{a_\theta\mathbf{u}_\theta}{R}\right)\mathbf{u}_R + \left(\operatorname{grad}a_\theta + \frac{a_R\mathbf{u}_\theta}{R} - \frac{a_\varphi\mathbf{u}_\varphi}{R\tan\theta}\right)\mathbf{u}_\theta$$

$$+ \left[\operatorname{grad}a_\varphi + \left(\frac{a_R}{R} + \frac{a_\theta}{R\tan\theta}\right)\mathbf{u}_\varphi\right]\mathbf{u}_\varphi$$

$$= \mathbf{u}_R\frac{\partial \mathbf{a}}{\partial R} + \mathbf{u}_\theta\frac{1}{R}\frac{\partial \mathbf{a}}{\partial \theta} + \mathbf{u}_\varphi\frac{1}{R\sin\theta}\frac{\partial \mathbf{a}}{\partial \varphi} \quad \text{(A4.86)}$$

$$\operatorname{div}\overline{\overline{a}} = \left(\operatorname{div}\mathbf{a}'_R - \frac{a_{\theta\theta} + a_{\varphi\varphi}}{R}\right)\mathbf{u}_R + \left(\operatorname{div}\mathbf{a}'_\theta + \frac{a_{\theta R}}{R} - \frac{a_{\varphi\varphi}}{R\tan\theta}\right)\mathbf{u}_\theta$$

$$+ \left(\operatorname{div}\mathbf{a}'_\varphi + \frac{a_{\varphi R}}{R} + \frac{a_{\varphi\theta}}{R\tan\theta}\right)\mathbf{u}_\varphi$$

$$= \frac{\partial \mathbf{a}_R}{\partial R} + \frac{2\mathbf{a}_R}{R} + \frac{1}{R}\frac{\partial \mathbf{a}_\theta}{\partial \theta} + \frac{\mathbf{a}_\theta}{R\tan\theta} + \frac{1}{R\sin\theta}\frac{\partial \mathbf{a}_\varphi}{\partial \varphi} \quad \text{(A4.87)}$$

$$\operatorname{curl} \overline{\overline{a}} = \left(\operatorname{curl} \mathbf{a}'_R + \frac{\mathbf{a}'_\theta \times \mathbf{u}_\theta}{R} + \frac{\mathbf{a}'_\varphi \times \mathbf{u}_\varphi}{R}\right)\mathbf{u}_R + \left(\operatorname{curl} \mathbf{a}'_\theta - \frac{\mathbf{a}'_R \times \mathbf{u}_\theta}{R} + \frac{\mathbf{a}'_\varphi \times \mathbf{u}_\theta}{R\tan\theta}\right)\mathbf{u}_\theta$$
$$+ \left(\operatorname{curl} \mathbf{a}'_\varphi - \frac{\mathbf{a}'_R \times \mathbf{u}_\varphi}{R} - \frac{\mathbf{a}'_\theta \times \mathbf{u}_\varphi}{R\tan\theta}\right)\mathbf{u}_\varphi$$
$$= \mathbf{u}_R\left(\frac{1}{R}\frac{\partial \mathbf{a}_\varphi}{\partial\theta} + \frac{\mathbf{a}_\varphi}{R\tan\theta} - \frac{1}{R\sin\theta}\frac{\partial \mathbf{a}_\theta}{\partial\varphi}\right)$$
$$+ \mathbf{u}_\theta\left(\frac{1}{R\sin\theta}\frac{\partial \mathbf{a}_R}{\partial\varphi} - \frac{\partial \mathbf{a}_\varphi}{\partial R} - \frac{\mathbf{a}_\varphi}{R}\right) + \mathbf{u}_\varphi\left(\frac{\partial \mathbf{a}_\theta}{\partial R} + \frac{\mathbf{a}_\theta}{R} - \frac{1}{R}\frac{\partial \mathbf{a}_R}{\partial\theta}\right). \tag{A4.88}$$

In particular:

$$\operatorname{grad} \mathbf{u}_R = \frac{\mathbf{u}_\theta\mathbf{u}_\theta}{R} + \frac{\mathbf{u}_\varphi\mathbf{u}_\varphi}{R} \tag{A4.89}$$

$$\operatorname{grad} \mathbf{u}_\theta = -\frac{\mathbf{u}_\theta\mathbf{u}_R}{R} + \frac{\mathbf{u}_\varphi\mathbf{u}_\varphi}{R\tan\theta} \tag{A4.90}$$

$$\operatorname{grad} \mathbf{u}_\varphi = -\frac{\mathbf{u}_\varphi\mathbf{u}_R}{R} - \frac{\mathbf{u}_\varphi\mathbf{u}_\theta}{R\tan\theta} \tag{A4.91}$$

$$\operatorname{grad}(R\mathbf{u}_R) = \overline{\overline{I}}. \tag{A4.92}$$

NOTES

In addition to [12, 165, 173] of the general bibliography:

I. V. Lindell, *Elements of Dyadic Algebra and Its Application in Electromagnetics*. Report S126, Radio Laboratory, Helsinki University of Technology, 1981.

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C. T. Tai, Some essential formulas in dyadic analysis and their applications. *Radio Sci.* **22**, 1283–1288, 1987.